PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 1, March 1972

REGULAR MODULES

DAVID J. FIELDHOUSE

ABSTRACT. Pierce [3] has shown that over a commutative regular ring every finitely generated submodule of an n-generated module is n-generated, and then has asked if this result holds for noncommutative regular rings. Here the result is shown for regular modules over any associative ring, which answers Pierce's question since a ring is regular if and only if every module is regular.

1. **Introduction.** Throughout this paper the word ring will mean an associative but not necessarily commutative ring with identity; all modules are unital.

If $n \geqq 1$ is an integer we will call a module *n-generated* iff it can be generated by n or fewer elements.

Pierce proves [3, 13.10] that over a commutative regular ring every finitely generated submodule of an n-generated module is n-generated, and then asks [1, p. 108, Problem 25] if this result holds for noncommutative regular rings.

We will prove this result for regular modules over any ring, which answers Pierce's question since a ring is regular iff every module is regular. Here regular ring means regular in the sense of von Neumann: for each $a \in A$ there exists $a' \in A$ with $a = aa'a$.

If $P \subseteq M$ are left A-modules then Cohn [1] calls P *pure* in M iff $0 \to X \otimes P \to X \otimes M$ is exact for all right A-modules X. We have called a module *regular* iff every submodule is pure. Basic properties of purity and regularity have been studied by the author in [2].

2. **The main theorem.**

THEOREM. *Suppose P is a pure submodule of the left A-module M. Then every finitely generated submodule of P is contained in an n-generated submodule of P for each integer n for which there exists an n-generated module between P and M, i.e., containing P and contained in M.*

PROOF. Suppose $K \subseteq P \subseteq N \subseteq M$ with K finitely generated and N n-generated. Then there exist finitely generated free modules E and F

Received by the editors June 1, 1971.

AMS 1970 *subject classifications*. Primary 16A30; Secondary 16A62, 16A64.

Key words and phrases. Regular module, regular ring, pure submodule, n-generated module.

with F n-generated, and onto maps $\varphi: E \to K$ and $\Psi: F \to N$ making the following diagram commute:

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & F \\ \varphi \downarrow & & \downarrow \Psi \\ K \subseteq & P & \subseteq N \end{array}$$

where the map α is induced by the projectivity of E.

Now P pure in M implies P pure in N. Hence there exists (see [**2**] and the Proposition below) a map $\sigma: F \to P$ such that $\sigma\alpha = z\varphi$ where $z: K \to P$ is the inclusion map. The image of F under σ is the desired n-generated module.

Corollary 1. *Provided there exists an n-generated module N with $P \subseteq N \subseteq M$ we have*:

(1) *P is the direct limit of n-generated modules.*

(2) *P finitely generated $\Rightarrow P$ is n-generated.*

Proof. (1) P is the direct limit of its finitely generated submodules.
(2) Take $K = P$.

Corollary 2. *If M is n-generated then every finitely generated submodule of P is contained in an n-generated submodule of P; in particular, P is n-generated if it is finitely generated.*

Corollary 3. *Every finitely generated submodule of an n-generated regular module is n-generated. Hence Pierce's question has an affirmative answer for any regular ring.*

Remark. In order to make this paper as selfcontained as possible, we present an independent proof of the existence of the map σ in the Theorem. It is also a slight generalization of the result given in (2).

Proposition. *If $P \subseteq M$ then P is pure in M iff given a commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & F \\ \varphi \downarrow & & \downarrow \Psi \\ P & \subseteq & M \end{array}$$

with E and F free and E finitely generated, then there exists a map $\sigma: F \to P$ such that $\sigma\alpha = \varphi$.

Proof. ($\Rightarrow$) Since αE is contained in some finitely generated free summand of F we can take F to be finitely generated free, without loss of generality. If (e_i) and (f_j) are bases for E and F respectively, and $\alpha e_i = \sum a_{ij} f_j$ then the commutativity of the square means that we have a finite system of equations: $\sum a_{ij}m_j=p_i \in P$, with $m_j=\Psi(f_j)$ and $p_i=\varphi(e_i)$. Since P is pure in M and the system is solvable in M, it is also solvable in P. If $p'_j \in P$ is a solution, define $\sigma(f_j)=p'_j$. It is now easy to verify that $\sigma\alpha=\varphi$.

($\Leftarrow$) Conversely if $\sum a_{ij}m_j=p_i \in P$ is a finite system of equations solvable in M then we can easily construct a commutative square of the above type. If (f_j) is a base for F then (σf_j) is a solution in P to the above system of equations.

References

1. P. M. Cohn, *On the free product of associative rings*, Math. Z. **71** (1959), 380–398. MR **21** #5648.

2. D. J. Fieldhouse, *Pure theories* Math. Ann. **184** (1969), 1–18. MR **40** #5699.

3. R. S. Pierce, *Modules over commutative regular rings*, Mem. Amer. Math. Soc. No. 70 (1967). MR **36** #151.

Department of Mathematics, University of Guelph, Guelph, Ontario, Canada