PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 1, March 1972

ON SELF-INJECTIVE GROUP RINGS

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ABSTRACT. Conditions are given under which the self-injectivity of the group ring AG implies the finiteness of G.

It has been known for some time that if A is a self-injective ring (associative with 1) and G is a finite group, then the group ring AG is self-injective; and conversely that if AG is self-injective then A is self-injective and G is locally finite [**4**]. Whether or not G must actually be finite, has been studied by Gentile [**2**] who obtained an affirmative answer in case A is a commutative ring which is torsion-free as a Z-module. His result is included in the following theorem (see Corollary). We denote the Jacobson radical of a ring A by Rad A, and use $o(H)$ to denote the order of a group H.

THEOREM. *If AG is a self-injective group ring and $o(H)$ is a unit in $A/\mathrm{Rad}\ A$ for all finite subgroups H of G, then (A is self-injective and) G is finite.*

PROOF. Since AG is self-injective, $AG/\mathrm{Rad}(AG)$ is self-injective and (Von Neumann) regular [**5**] and similarly since AG self-injective$\Rightarrow A$ self-injective, it follows that $A/\mathrm{Rad}\ A$ is regular.

Since G is locally finite, $\mathrm{Rad}\ A=\mathrm{Rad}(AG)\cap A$ [**1**] so $(\mathrm{Rad}\ A)G\subseteq \mathrm{Rad}(AG)$ and therefore

$$AG/\mathrm{Rad}\ (AG)\simeq\frac{AG/(\mathrm{Rad}\ A)G}{\mathrm{Rad}(AG)/(\mathrm{Rad}\ A)G}\simeq\frac{(A/\mathrm{Rad}\ A)G}{\mathrm{Rad}((A/\mathrm{Rad}\ A)G)}.$$

Since $o(H)$ is a unit in the regular ring $A/\mathrm{Rad}\ A$ for all finite subgroups H of G, it follows that $(A/\mathrm{Rad}\ A)G$ is regular [**1**], and therefore has zero radical. Thus $AG/\mathrm{Rad}(AG)\simeq(A/\mathrm{Rad}\ A)G$ and it suffices to consider a regular self-injective group ring AG.

It is easily shown that when AG is self-injective, so is AH for all subgroups H of G (see e.g. [**2**]), so without loss of generality we assume G is countable. Then the fundamental ideal Δ of AG is countably generated, and since AG is regular, a result of Kaplansky [**3**] shows that Δ is

Received by the editors April 26, 1971.
AMS 1970 *subject classifications*. Primary 16A26, 16A52; Secondary 16A30.
Key words and phrases. Group ring, self-injective ring, regular ring.

projective. Thus Δ is (isomorphic to) a direct summand of a free AG-module F. Suppose $F=\Delta\oplus K$ where $K\neq 0$. Since F is isomorphic to a direct sum of copies of AG, it has a canonical multiplication. Let $\mathrm{Ann}_F\,\Delta$ be the left annihilator of Δ in F. Then $K\Delta\subset K\cap\Delta=0$ so $\mathrm{Ann}_F\,\Delta\neq 0$. Let $x=\sum_{i=1}^{n} a_i g_i$ be any nonzero entry from a nonzero element $y=(x_j)$ of $\mathrm{Ann}_F\,\Delta\subset\bigoplus AG$. If G is infinite, we can choose an $h\in G$ such that $h\neq g_1^{-1}g_i$ for all $i=1,\cdots,n$. Then $y(1-h)\neq 0$ which is a contradiction, so G must be finite.

It remains to consider the case when $K=0$. Then Δ is a free AG-module so there exists an isomorphism $\varphi:\Delta\to\bigoplus AG$. Let p and i be canonical homomorphisms $p:\bigoplus AG\to AG$ and $i:AG\to\bigoplus AG$ so that pi is the identity on AG. Now $p\varphi:\Delta\to AG$ is an AG-module homomorphism so by the self-injectivity of AG, there exists an $r\in AG$ such that $p\varphi(x)=rx$ for all $x\in\Delta$. Then

$$\varphi^{-1}ip\varphi(x) = \varphi^{-1}i(rx) = r\varphi^{-1}i(x) \quad \text{for all } x\in\Delta$$

and

$$x = p\varphi\varphi^{-1}i(x) = r\varphi^{-1}i(x) \quad \text{for all } x\in AG$$

so

$$\varphi^{-1}ip\varphi(x) = x \quad \text{for all } x\in\Delta.$$

Therefore ip is the identity on $\bigoplus AG$; so p is an isomorphism and $\Delta\simeq AG$ (as AG-modules). Then Δ is AG-injective, so is a direct summand of AG, whence G is finite [**1**].

COROLLARY (GENTILE). *If A is commutative and torsion-free as a Z-module, then AG self-injective $\Rightarrow G$ finite.*

PROOF. A contains the rationals [**2**] so every integer is a unit in A and therefore in $A/\mathrm{Rad}\,A$.

ADDED IN PROOF. Recently G. Renault (*Sur les anneaux de groupes*, C. R. Acad. Sci. Paris Ser. A **273** (1971), 84–87) has shown that AG self-injective implies G finite.

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