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# BOUNDED LIMITS OF ANALYTIC FUNCTIONS[1]

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ABSTRACT. Let $U$ be a bounded open plane set, and let $f$ be a bounded analytic function on $U$, which is the pointwise limit of a bounded sequence $\{f_n\}$ of uniformly continuous analytic functions. It is shown that one can find another such sequence $\{f_n'\}$, converging to $f$, and bounded by the supremum norm of $f$. A similar result is proved for approximation by rational functions.

In this paper the following problem is considered: let $U$ be a bounded open subset of the complex plane $C$, and let $A$ be a set of bounded analytic functions on $U$. Which bounded analytic functions on $U$ are limits of bounded sequences of functions in $A$ converging pointwise in $U$?

In the case where $A$ consists of the polynomials this question was settled by Rubel and Shields [**4**], a special case had earlier been treated by Farrell [**2**]. A function $f$ on $U$ is the pointwise limit of some bounded sequence of polynomials if and only if it is the restriction to $U$ of a bounded analytic function on $U^*$, the Carathéodory hull of $U$. ($U^*$ is the interior of the complement of the unbounded component of the complement of the closure of $U$.)

Rubel and Shields gave an example to show that the set of such pointwise limits need not be closed under uniform convergence. They constructed a set $U$ and a sequence $\{f_n\}$ of bounded analytic functions converging uniformly on $U$ to $f$, such that each $f_n$ is a pointwise bounded limit of polynomials, but that the bounds on the approximating sequences of polynomials necessarily tended to infinity with $n$, so that no bounded diagonal subsequence converging to $f$ could be found. The object of this paper is to show that if $A$ is either the algebra of uniformly continuous analytic functions on $U$ or (under mild hypotheses on $\partial U$) the rational functions with poles outside $\bar{U}$, then this phenomenon cannot occur.

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*Notation.* $A(U)$ denotes the algebra of uniformly continuous analytic functions on the bounded open set $U$ (which we regard as continuous

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functions on the closure $\bar{U}$). For a compact plane set $X$, we denote by $R(X)$ the uniform closure on $X$ of the rational functions with poles outside $X$. $X^0$ and $\partial X$ denote the interior and boundary of $X$ respectively. "Measure" means "complex Borel measure" and $A(U)^\perp$ is the set of measures $\mu$ on $\bar{U}$ such that $\int f\,d\mu=0$ for all $f\in A(U)$. The symbol $\|f\|$ means the supremum of $|f|$ over the domain of definition of $f$, and $\|f\|_S$ means the supremum of $|f|$ over the set $S$. Finally $H^\infty(U)$ denotes the algebra of all bounded analytic functions on $U$.

Fix a bounded open set $U$, and let $B=\{f\in H^\infty(U)$: there exists a sequence $\{f_n\}$ in $A(U)$ with $\sup_n\|f_n\|<\infty$ and $f_n\to f$ pointwise in $U\}$.

THEOREM 1. *Let $f\in B$. Then we can find a sequence $\{f_n\}$, with $f_n\in A(U)$, $\|f_n\|\leqq\|f\|$, converging to $f$ pointwise on $U$.*

We divide the proof into two steps.

*Step* 1. Assume the theorem is false and let $\varepsilon>0$. Then we can find a sequence $\{g_n\}$ in $A(U)$, a set $F\subseteq\bar{U}$, and a measure $\sigma\in A(U)^\perp$, such that $\|g_n\|\leqq 1$, $g_n\to g$ pointwise on $U$, and $\|g\|_U<\varepsilon$, $|1-g_n|<\varepsilon$ on $F$, and $\sigma(F)\neq 0$.

PROOF. Assuming the theorem false we can find $f\in B$ with $\|f\|=1$ such that, if we define

$$\lambda = \inf\{\sup_n \|f_n\| : \{f_n\} \text{ is a sequence in } A(U) \text{ with } f_n \to f \text{ pointwise in } U\},$$

then $\lambda>1$.

Let $\{f_n\}$ be a sequence in $A(U)$ with $\|f_n\|\leqq\lambda$ and $f_n\to f$ pointwise. Let $m$ be a positive integer such that $\lambda^{-m}<\varepsilon$ and let $\eta=\min(\lambda-1, \lambda\varepsilon/2m)$. By the definition of $\lambda$ there is a compact set $K\subseteq U$ such that $f$ is not in the closure of $T=\{h\in A(U):\|h\|\leqq\lambda-\eta\}$ in the topology of uniform convergence on $K$. Thus we can find a measure $\mu$ on $K$ such that $|\int h\,d\mu|\leqq 1$ for $h\in T$ but $|\int f\,d\mu|>1$. The functional $h\to\int h\,d\mu$ on $A(U)$ has norm $\leqq(\lambda-\eta)^{-1}$, and has a norm-preserving extension to $C(\bar{U})$ represented by a measure $\nu$, $\|\nu\|\leqq(\lambda-\eta)^{-1}$.

Then $\sigma=\mu-\nu\in A(U)^\perp$. Let $G$ be a cluster point of $\{f_n\}$ in $L^\infty(|\sigma|+|\mu|)$. Then $\int G\,d\sigma=0$ and so $1<|\int f\,d\mu|=\lim_n|\int f_n\,d\mu|=|\int G\,d\mu|=|\int G\,d\nu|\leqq\|G\|\,\|\nu\|\leqq(\lambda-\eta)^{-1}\|G\|$. Hence $\|G\|>\lambda-\eta>1$. Clearly $\|G\|\leqq\lambda$, and $|G|\leqq 1$ a.e. ($|\mu|$), so that $\lambda-\eta<|G|\leqq\lambda$ on some set $F_1$ with $|\sigma|\,(F_1)>0$.

The annulus $\{\zeta:\lambda-\eta\leqq|\zeta|\leqq\lambda\}$ can be covered by finitely many discs with centers on $\{|\zeta|=\lambda\}$ and radii $2\eta$. The inverse of one such disc under $G$ has positive $|\sigma|$-measure; multiplying $G$ (and $f_n$ and $f$) by a constant of modulus 1 we may assume that the center is $\lambda$.

Thus we can find a compact set $F_2\subseteq F_1$ with $|\sigma|(F_2)>0$ and $|\lambda-G|<2\eta$ on $F_2$.

By passing to a subsequence we may assume $f_n \to G$ weak* in $L^\infty(|\sigma|)$, hence weakly in $L^2(|\sigma|)$. Then we can find a sequence $\{f'_k\}$, where each $f'_k$ is a convex combination of functions $f_n$, with $f'_k \to G$ in norm in $L^2(|\sigma|)$, and still $f'_k \to f$ in $U$. Again passing to a subsequence we may assume $f'_k \to G$ a.e. ($|\sigma|$). By Egoroff's theorem we can find a compact set $F \subseteq F_2$ with $\sigma(F) \neq 0$ such that $f'_k \to G$ uniformly on $F$. Then we can find $k_0$ so that for $k > k_0$, $|\lambda - f'_k| < 2\eta$ on $F$.

Put $g_k = (\lambda^{-1} f'_k)^m$ and $g = (\lambda^{-1} f)^m$. Then $g_k \in A(U)$, $g_k \to g$ pointwise in $U$, and $\|g\| < \varepsilon$.

Finally, on $F$ we have $|1 - g_k| = |1 - (\lambda^{-1} f'_k)^m| < \varepsilon$ by the choice of $\eta$, for $k > k_0$. This completes Step 1.

*Step* 2. To prove the theorem we show that Step 1 leads to a contradiction if $\varepsilon$ is small enough. The proof is based on a construction due to Øksendal [**5**, Lemma 2.1], and a variation of Vituskin's $T_\varphi$ technique due to Gamelin.

We use $A_1, A_2, \cdots$ to denote absolute constants. Fix $\delta > 0$, and choose discs $\Delta_1, \cdots, \Delta_r$ with centers $z_1, \cdots, z_r$ and radius $\delta$, covering $F$, and continuously differentiable functions $\varphi_1, \cdots, \varphi_r$ such that:

(i) at most 25 of the discs $\Delta_i$ meet any given point, each $\Delta_i$ meets $F$;

(ii) $\varphi_i$ vanishes outside a compact subset of $\Delta_i$, $0 \leqq \varphi_i \leqq 1$, $\sum_{i=1}^r \varphi_i = 1$ on a neighborhood of $F$, and $|\text{grad}\,\varphi_i| \leqq A_1 \delta^{-1}$. (For details of this construction see [**3**, VIII.7.1].) For $i = 1, 2, \cdots, r$ and $n = 1, 2, \cdots$, define

$$h_i^{(n)}(\zeta) = \frac{1}{\pi} \int_U \frac{\partial \varphi_i}{\partial \bar{z}} \frac{g_n(z)}{z - \zeta}\, dm(z), \qquad \zeta \in \boldsymbol{C},$$

where $m$ denotes Lebesgue measure, and $h_i$ similarly with $g_n$ replaced by $g$. Then $h_i^{(n)} \to h_i$ uniformly, since

$$|h_i^{(n)}(\zeta) - h_i(\zeta)| \leqq \frac{A_1}{\pi\delta} \int_U \left| \frac{1}{z - \zeta} \right| |g_n(z) - g(z)|\, dm(z)$$

$$\leqq \frac{A_1}{\pi\delta} \left( \int_U \frac{1}{|z - \zeta|^{3/2}} \right)^{2/3} \left( \int_U |g_n(z) - g(z)|^3\, dm(z) \right)^{1/3}$$

the first integral is bounded by a fixed constant and the second tends to zero.

We have $\|h_i\| \leqq A_2 \|g\| < A_2 \varepsilon$, so if $n = n(\delta)$ is chosen large enough we have $\|h_i^{(n)}\| < A_2 \varepsilon$. Since $h_i$ is analytic outside $\Delta_i$ and vanishes at $\infty$, in fact we have

$$|h_i^{(n)}(\zeta)| < A_2 \varepsilon \min\left(1, \frac{\delta}{|\zeta - z_i|}\right).$$

We observe that $g_n\varphi_i+h_i^{(n)}\in A(U)$ since

$$\frac{1}{\pi}\int_{\Delta_i\setminus U}\frac{\partial\varphi_i}{\partial\bar z}\frac{g_n(z)}{z-\zeta}\,dm(z) \quad\text{and}\quad g_n\varphi_i+\frac{1}{\pi}\int_{\Delta_i}\frac{\partial\varphi_i}{\partial\bar z}\frac{g_n(z)}{z-\zeta}\,dm(z)$$

are both in $A(U)$ (see [3, VIII, 7.1]). Define

$$H_\delta=\sum_{i=1}^{r}(g_n\varphi_i+h_i^{(n)})^3\in A(U).$$

We assert

(1) $|H_\delta(\zeta)|\leqq A_3\min(1,\delta/d(\zeta,F)),\ \zeta\in\bar U$.

(2) If $\zeta\in F$, $|1-H_\delta(\zeta)|<A_4<1$ provided $\varepsilon<A_5$. To prove this write

$$\begin{aligned}H_\delta(\zeta)&=g_n^3\sum_{i=1}^{r}\varphi_i^3+3g_n\sum_{i=1}^{r}\varphi_ih_i^{(n)}(g_n\varphi_i+h_i^{(n)})+\sum_{i=1}^{r}(h_i^{(n)})^3\\&=T_1+T_2+T_3\quad\text{say.}\end{aligned}$$

We first estimate $T_3$:

$$\begin{aligned}|T_3|&\leqq A_2^3\varepsilon^3\sum_{i=1}^{r}\min\left(1,\frac{\delta^3}{|\zeta-z_i|^3}\right)\\&\leqq A_6\varepsilon^3\min\left(1,\frac{\delta}{d(\zeta,F)}\right)\end{aligned}$$

by an easy calculation, along the lines of [3, p. 212]. For any fixed $\zeta$, at most 25 of the terms summed in $T_1$ and $T_2$ are nonzero, so (1) follows.

For (2) observe that $\|T_2\|\leqq 75A_2\varepsilon(1+A_2\varepsilon)$, so $\|T_2+T_3\|<A_7\varepsilon$. Let $\psi=\sum_{i=1}^{r}\varphi_i^3$, since $\sum\varphi_i=1$ on $F$, if $\zeta\in F$ then $\varphi_i(\zeta)\geqq 25^{-1}$ for some $i$, so $\psi(\zeta)\geqq 25^{-3}$. Thus, for $\zeta\in F$,

$$\begin{aligned}|1-H_\delta(\zeta)|&\leqq A_7\varepsilon+|1-g_n^3(\zeta)\psi(\zeta)|\\&\leqq A_7\varepsilon+(1-\psi(\zeta))+\psi(\zeta)\,|1-g_n^3(\zeta)|\\&\leqq 1-25^{-3}+(3+A_7)\varepsilon\\&\leqq 1-\frac{25^{-3}}{2}\end{aligned}$$

provided

$$\varepsilon<\frac{1}{2.25^3(3+A_7)},$$

which is (2).

Thus as $\delta\to 0$, by (1) $H_\delta\to 0$ boundedly on $\bar U\setminus F$. Hence for each integer $k$

we can choose $\delta_k > 0$ so that

$$\left| \int_{\bar{U} \setminus F} 1 - (1 - H_{\delta_k})^k \, d\sigma \right| < \frac{1}{k}.$$

Since $\sigma \in A(U)^\perp$, $\int_{\bar{U}} 1-(1-H_{\delta_k})^k \, d\sigma = 0$ and so

$$\left| \int_F 1 - (1 - H_{\delta_k})^k \, d\sigma \right| < \frac{1}{k}.$$

As $k \to \infty$, the integrand tends uniformly to 1 on $F$, hence $\sigma(F) = 0$, a contradiction. Theorem 1 is proved.

COROLLARY. *$B$ is closed under pointwise bounded convergence.*

Next we prove an analogous result for $R(X)$. Let $\tau$ denote plane Lebesgue measure restricted to the points of $K$ which are not peak points for $R(X)$. Let $B_R$ denote the set of $f \in L^\infty(\tau)$ for which there exists a sequence $f_n \in R(X)$ with $\sup_n \|f_n\| < \infty$ and $f_n \to f$ weak* in $L^\infty(\tau)$.

THEOREM 2. *Let $f \in B_R$. We can find a sequence $\{f_n\}$ in $R(X)$ with $\|f_n\| \leqq \|f\|_\infty$ and $f_n \to f$ weak* in $L^\infty(\tau)$.*

PROOF. This is essentially the same as that of Theorem 1 so we merely indicate the necessary modifications. $R(X)$ replaces $A(U)$ and $X$ replaces $\bar{U}$. $\mu$ is now a measure absolutely continuous with respect to $\tau$. Everything goes through until the construction of $h_i^{(n)}$, now we define

$$h_i^{(n)}(\zeta) = \frac{1}{\pi} \int \frac{\partial \varphi_i}{\partial \bar{z}} \frac{g_n(z)}{z - \zeta} \, d\tau(z).$$

The only problem is to show $g_n \varphi_i + h_i^{(n)} \in R(X)$, i.e. that

$$F(\zeta) = \int_\partial \frac{\partial \varphi_i}{\partial \bar{z}} \frac{g_n(z)}{z - \zeta} \, dm(z)$$

is in $R(X)$, where $\partial$ is the set of peak points. But if $\theta \perp R(X)$ then

$$\begin{aligned} \int_X F(\zeta) \, d\theta(\zeta) &= \int_\partial \int_X \frac{\partial \varphi}{\partial \bar{z}} \frac{g_n(z)}{z - \zeta} \, d\theta(\zeta) \, dm(z) \\ &= \int_\partial \frac{\partial \varphi}{\partial \bar{z}} g_n(z) \left( \int_X \frac{1}{z - \zeta} \, d\theta(\zeta) \right) dm(z) \end{aligned}$$

the integrals converging absolutely since $1/|z|$ is integrable over any bounded set. If $z \in X$ is such that $\int (1/|z-\zeta|)|d\theta(\zeta)| < \infty$ and $\int (1/(z-\zeta)) \, d\theta(\zeta) \neq 0$ then $z$ is not a peak point for $R(X)$ by the proof of

Theorem II.8.5 of [3]. Hence $\int_X (1/(z-\zeta))\, d\theta(\zeta)=0$ for almost all $z\in\partial$, so that $\int F\, d\theta=0$. Hence $F\in R(X)$.

The rest of the proof works as before.

*Notes.* (1) If $\{f_n\}$ is a bounded sequence in $R(X)$ converging weak* in $L^\infty(\tau)$ to $f\in B_R$, then in fact $f_n$ converges pointwise on $X\backslash\partial$ to $g$ say, where $g=f$ a.e. $(\tau)$ and $g$ depends only on $f$ (not on the choice of $\{f_n\}$). To see this choose a sequence $\{g_n\}$ of convex combinations of the functions $f_n$ converging a.e. $(\tau)$ to $f$. Let $z\in X\backslash\partial$, let $\varepsilon>0$ be given, and choose $\zeta\in K\backslash\partial$ so that $g_n(\zeta)\to f(\zeta)$ and $\|\tilde{z}-\tilde{\zeta}\|<\varepsilon$ where $\tilde{z}$, $\tilde{\zeta}$ are the evaluation functionals on $R(X)$. (This is possible by [**1**, Theorem 2].) Then $|g_n(\zeta)-g_n(z)|<M\varepsilon$ where $M=\sup_n\|f_n\|$. Since $\varepsilon$ is arbitrary we deduce that $g_n(z)$ converges to a limit $g(z)$ such that $|g(z)-f(\zeta)|<M\varepsilon$ for almost all $\zeta$ satisfying $\|\tilde{z}-\tilde{\zeta}\|<\varepsilon$. Then $g_n$ converges pointwise to $g$ on $X\backslash\partial$ and $g$ is determined by $f$. It follows that the original sequence $\{f_n\}$ converges pointwise to $g$ on $X\backslash\partial$.

(2) The question naturally arises as to whether one can obtain a similar result involving convergence only on $X^0$. If almost all points of $\partial X$ are peak points then Theorem 2 answers this question. On the other hand one can easily construct examples to show that some restriction is necessary. In general we have the following: let $f\in H^\infty(X^0)$ be the limit of a bounded sequence $\{f_n\}$ in $R(X)$. Then we can find a subsequence converging pointwise on $X\backslash\partial$ to $g$ say, so that $g|X^0=f$. If $R(\partial K)=C(\partial K)$ then $g$ depends only on $f$ (for if $f=0$ then $f_n\to 0$ pointwise in $X^0$, so

$$\frac{f_n(z)-f_n(z_0)}{z-z_0}\to\frac{g(z)}{z-z_0}\in B_R$$

for each $z_0\in X^0$, whence $hg\in B_R$ for all $h\in C(X)$, which implies $g=0$ on $X\backslash\partial$ in view of the inequality $|g(z)-g(\zeta)|\leqq M\|\tilde{z}-\tilde{\zeta}\|$, $z,\zeta\in X\backslash\partial$, $M=\sup_n\|f_n\|$). We conjecture that in this case $\|g\|=\|f\|$, which is equivalent to the analogue of Theorem 2 for convergence on $X^0$. A more concrete way of stating this conjecture is as follows: suppose $R(\partial X)=C(\partial X)$. There exists $\varepsilon>0$ such that if $\{f_n\}$ is a sequence in $R(X)$ with $\|f_n\|\leqq 1, f_n\to f$ on $X^0$ with $\|f\|<\varepsilon$, and $z$ is a point in $\partial X$ with $|1-f_n(z)|<\varepsilon$ for each $n$, then $z$ is a peak point for $R(X)$.

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