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ON THE RADIUS OF CONVEXITY AND STARLIKENESS OF UNIVALENT FUNCTIONS

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ABSTRACT. In this paper, the converses of the theorems of Bernardi (Trans. Amer. Math. Soc. **135** (1969), 429–446) for the subclasses of univalent functions, namely, starlike functions of order β, convex functions of order β and close-to-convex functions of type β and order λ have been derived. In particular, these results are sharp and contain the theorems of Padmanabhan (J. London Math. Soc. (2) **1** (1969), 226–231) and Bernardi (Proc. Amer. Math. Soc. **24** (1970), 312–318) as special cases.

1. **Introduction.** In this paper we study some classes of univalent functions. By S we denote the class of functions $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$, which are regular and univalent in the unit disc $D\{|z|<1\}$, while S^* denotes the class of functions in S which map D onto a starlike region with respect to the origin. An equivalent analytic characterization for functions of S^* is well known [**8**]. By S_β^* we denote the class of functions $f(z)$ in S^* having the additional property

$$\text{(1.1)} \qquad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq \beta; \qquad z \in D; \quad 0 \leqq \beta \leqq 1.$$

Here β is referred as the order of starlike functions $f(z)$ and we identify $S_0^* \equiv S^*$. The class of functions $f(z)$, which are in S and map D onto a convex domain, is denoted by C, while C_β denotes the class of univalent functions of order β, if

$$\text{(1.2)} \qquad \operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geqq \beta; \qquad z \in D, \quad 0 \leqq \beta \leqq 1.$$

If $f(z)\in S$ and $g(z)\in S_\beta^*$ satisfy the condition

$$\text{(1.3)} \qquad \operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} \geqq \lambda; \qquad z \in D, \quad 0 \leqq \lambda \leqq 1,$$

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then $f(z)$ is said to be a close-to-convex function of order λ and type β. We denote this class by $\Gamma(\lambda, \beta)$. If $\lambda=0=\beta$, then $f(z)$ is simply said to be a close-to-convex function with respect to the function $g(z)$ and the class $\Gamma(0, 0)$ is identified by Γ. This concept of close-to-convex functions is due to Kaplan [3] and its extension appears in the works of Robertson, Libera and others.

Libera [4] in 1965 established the following theorems:

THEOREM A [LIBERA]. *If $f\in S^*$ (or $f\in C$) then the function $F(z)=(2/z)\int_0^z f(t)\,dt\in S^*$ (or $F\in C$).*

THEOREM B [LIBERA]. *If $f\in\Gamma$ with respect to $g(z)$ and $F(z)=(2/z)\int_0^z f(t)\,dt$ and $G(z)=(2/z)\int_0^z g(t)\,dt$ then $F\in\Gamma$ with respect to G.*

Bernardi [1] extended Theorems A and B, and proved the following:

THEOREM C [BERNARDI]. *If $f(z)\in S^*$ (or C), $c=1, 2, 3, \cdots$,*

$$g(z)=\sum_{n=1}^{\infty}\left(\frac{c+1}{c+n}\right)a_n z^n=(c+1)z^{-c}\int_0^z t^{c-1}f(t)\,dt \qquad (a_1=1)$$

then $g(z)\in S^*$ (or C).

THEOREM D [BERNARDI]. *If $f\in\Gamma$ with respect to g and*

$$F(z)=(c+1)z^{-c}\int_0^z t^{c-1}f(t)\,dt; \qquad G(z)=(c+1)z^{-c}\int_0^z t^{c-1}g(t)\,dt,$$

$c=1, 2, 3, \cdots$, then $F\in\Gamma$ with respect to G.

The converse problem of Libera [4] was treated by Livingston [5] who proved the following:

THEOREM E [LIVINGSTON]. *If $F\in S^*$ then $f(z)=\frac{1}{2}[zF(z)]'$ is starlike for $|z|<\frac{1}{2}$. This result is sharp.*

THEOREM F [LIVINGSTON]. *If F is in C, then $f(z)=\frac{1}{2}[zF(z)]'$ is univalent in D and is convex for $|z|<\frac{1}{2}$. This result is sharp.*

Bernardi [2] again considered the converse problem of Theorems C and D and obtained the following:

THEOREM G [BERNARDI]. *If $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$,*

$$g(z)=\sum_{n=1}^{\infty}\left(\frac{c+1}{c+n}\right)a_n z^n=(c+1)z^{-c}\int_0^z t^{c-1}f(t)\,dt$$

with $a_1=1$ and $c=1, 2, 3, \cdots$ and $g(z)\in S^*$ (or C) then $f(z)$ is starlike (or convex) in the region $|z|<(-2+(3+c^2)^{1/2})/(c-1)$ for $c=2, 3, 4, \cdots$ and $|z|<\frac{1}{2}$ for $c=1$. This result is sharp.

THEOREM H [BERNARDI]. *Let $F(z)$ be close-to-convex with respect to $G(z)\in S^*$ and*

$$f(z) = \frac{1}{1+c}\, z^{1-c}[z^c F(z)]', \qquad g(z) = \frac{1}{1+c}\, z^{1-c}[z^c G(z)]'$$

then $f(z)$ is close-to-convex with respect to $g(z)$ in the region

$$|z| < (-2 + (3 + c^2)^{1/2})/(c-1)$$

for $c=2, 3, 4, \cdots$ and $|z|<\frac{1}{2}$ if $c=1$.

Padmanabhan [7] considered the converse problem of Libera [4] for the class S^*_β, C_β and $\Gamma(\lambda, \beta)$, where $0\leqq\beta\leqq\frac{1}{2}$. In this paper we are mainly concerned with radius of starlikeness and radius of convexity for functions in S^*_β, C_β and $\Gamma(\lambda, \beta)$, respectively. In particular, we derive the converses of Theorems G and H of Bernardi for the classes S^*_β, $\Gamma(\lambda, \beta)$ and C_β. We notice that these results are sharp. With these extensions we deduce the theorems of Padmanabhan [7] also. Incidently, the proof of our Theorem 1, which includes a theorem of Padmanabhan, is much simpler and can also be adopted for the restricted case considered by him. We state here a lemma due to Bernardi [1] which we shall need,

LEMMA [1, p. 430]. *Let $f(z)$ and $g(z)$ be regular in $|z|<1$, $g(z)$ map $|z|<1$ onto a many sheeted starlike region, α, β real,*

$$\operatorname{Re}\left\{e^{i\beta}\frac{f'(z)}{g'(z)}\right\} > \alpha$$

for $|z|<1$, $g(0)=f(0)=0$. Then $\operatorname{Re}\{e^{i\beta}f(z)/g(z)\}>\alpha$ for $|z|<1$.

Further, if $f(z)\in S^*$, $g(z)=\int_0^z H(t)\,dt=\int_0^z t^{p-1}f(t)\,dt$, then $g(z)$ is $(p+1)$-valent starlike for $p=1, 2, 3, \cdots$.

2. We shall prove the following:

THEOREM 1. *If $f(z)=z+\sum_{n=2}^\infty a_n z^n\in S^*_\beta$ and*

$$g(z) = \sum_{n=1}^\infty \left(\frac{c+1}{c+n}\right) a_n z^n = (c+1)z^{-c}\int_0^z t^{c-1}f(t)\, d ,$$

with $a_1=1$ and $c=1, 2, 3, \cdots$, then $g(z)\in S^*_\beta$ and conversely, if $g(z)\in S^*_\beta$ then $f(z)$ is starlike of order β in the region

$$|z| < r_0 = \frac{-(2-\beta) + (3+\beta^2+c^2+2c\beta-2\beta)^{1/2}}{c+2\beta-1}, \qquad \text{if } c = 2, 3, \ldots,$$

$$= \tfrac{1}{2}, \qquad \text{if } c = 1 \text{ and } \beta = 0,$$

$$= \frac{-(2-\beta)+(4+\beta^2)^{1/2}}{2\beta}, \qquad \text{if } c = 1 \text{ and } 0 < \beta < 1.$$

PROOF. If $J(z)=\int_0^z t^{c-1}f(t)\,dt$ then it easily follows on lines similar to those given by Bernardi [**1**, p. 431] that

$$\text{Re}\left\{\frac{[z^{c+1}g'(z)]'}{J'(z)}\right\} = (c+1)\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq (c+1)\beta. \tag{2.1}$$

From (2.1) and the lemma, the first part of the theorem follows.

Conversely, by the hypothesis of the theorem we have

$$\frac{zg'(z)}{g(z)} = \frac{zJ' - cJ}{J}. \tag{2.2}$$

Further, since $g(z)$ is starlike function of order β, so there exists a function $\omega(z)$ which is regular in the unit disc D and satisfies the conditions of Schwarz's lemma, such that

$$\frac{zg'(z)}{g(z)} = \frac{1-(1-2\beta)\omega(z)}{1+\omega(z)}. \tag{2.3}$$

From (2.2) and (2.3) it follows that

$$f(z) = \frac{[(1+c)+\{c+(2\beta-1)\}\omega(z)]J}{[1+\omega(z)]z^c}.$$

Differentiating $f(z)$ logarithmically and simplifying finally we get

$$\frac{zf'(z)}{f(z)} - \beta = (1-\beta) \times \left[\frac{1-\omega(z)}{1+\omega(z)} - \frac{2z\omega'(z)}{[1+\omega(z)][(1+c)+(c+2\beta-1)\omega(z)]}\right]. \tag{2.4}$$

But

$$\text{Re}\left\{\frac{1-\omega(z)}{1+\omega(z)}\right\} = \frac{1-|\omega(z)|^2}{|1+\omega(z)|^2} \tag{2.5}$$

and

$$\text{Re}\left\{\frac{2z\omega'(z)}{[1+\omega(z)][(1+c)+(c+2\beta-1)\omega(z)]}\right\} \leqq \frac{2|z|(1-|\omega(z)|^2)}{(1-|z|^2)\,|1+\omega(z)|\,|(1+c)+(c+2\beta-1)\omega(z)|}. \tag{2.6}$$

The last inequality is obtained by using the following well-known inequality [**6**, p. 168]:

$$|\omega'(z)| \leqq (1-|\omega(z)|^2)/(1-|z|^2). \tag{2.7}$$

From (2.4) thru (2.7), we note that $f(z)$ is starlike of order β, if

$$\frac{2|z|(1-|\omega(z)|^2)}{|1+\omega(z)|(1-|z|^2)|(1+c)+(c+2\beta-1)\omega(z)|} \leqq \frac{1-|\omega(z)|^2}{|1+\omega(z)|^2}$$

or

$$(2.8)\qquad \frac{2|z|}{1-|z|^2} \leqq (1+c)\left|1+\frac{c+2\beta-1}{1+c}\omega(z)\right| \Big/ |1+\omega(z)|.$$

Since $|\omega(z)| \leqq |z|$ and $(c+2\beta-1)/(1+c) \leqq 1$, we have

$$(2.9)\qquad 1+\frac{c+2\beta-1}{1+c}|z| \Big/ (1+|z|) \leqq \left|1+\frac{c+2\beta-1}{1+c}\omega(z)\right| \Big/ |1+\omega(z)|.$$

Hence, by (2.8) and (2.9), we obtain that $f(z) \in S_\beta^*$ if

$$2|z| \leqq \{(1+c)+(c+2\beta-1)|z|\}(1-|z|)$$

i.e., if

$$(1+c)-2(2-\beta)|z|-(c+2\beta-1)|z|^2 > 0.$$

Let $P(|z|)=P(r)=(1+c)-2(2-\beta)r-(c+2\beta-1)r^2$. Since $P(0)=1+c$ and $P'(r)<0$, the positive root r_0 for which $P(r)>0$ must be less than the root of the polynomial $P(r)=0$. This gives the required value of r_0 and the proof of Theorem 1 is complete.

The following example shows that the result of Theorem 1 is sharp for each c.

EXAMPLE 1. *Consider the function*

$$g(z)=z(1-z)^{-2(1-\beta)}; \qquad 0 \leqq \beta \leqq 1.$$

Clearly $g(z) \in S_\beta^*$ and $f(z)=(z^{1-c}/(1+c))[z^c g(z)]'$ imply that

$$\frac{zf'(z)}{f(z)}-\beta=\frac{(1-\beta)[(1+c)+2(2-\beta)z-(c+2\beta-1)z^2]}{(1-z)[(1+c)-(c+2\beta-1)z]}.$$

Thus $zf'(z)/f(z)-\beta=0$ for $z=-r_0$. Hence $f(z)$ is not starlike in any disc $|z|<r$, if $r>r_0$.

Theorem G of Bernardi now follows by taking $c=1, 2, 3, \cdots$ and $\beta=0$.

If we take $c=1$ and $0 \leqq \beta \leqq \frac{1}{2}$, then the following theorem of Padmanabhan [7] follows as a corollary to Theorem 1.

THEOREM [PADMANABHAN]. *Let* $g(z) \in S_\beta^*$. *Then* $f(z)=\frac{1}{2}[zg(z)]'$ *is starlike of order* β *for*

$$|z| < \frac{(\beta-2)+(\beta^2+4)^{1/2}}{2\beta}; \qquad 0 \leqq \beta \leqq \tfrac{1}{2}.$$

THEOREM 2. *If* $g(z)\in C_\beta$, $f(z)=(1/(1+c))z^{1-c}[z^c g(z)]'$, $c=1, 2, 3, \cdots$, *then* $f(z)$ *is convex of order* β *in* $|z|<r_0$ *where* r_0 *is defined as in Theorem* 1. *The result is sharp.*

PROOF. Proof of Theorem 2 follows immediately by using the fact that if $g(z)\in C_\beta$ then $zg'(z)\in S_\beta^*$ and conversely (see [**8**]).

The following example shows that the result of Theorem 2 is sharp.

EXAMPLE 2. *Let* $g(z)=1-(1-z)^{2\beta-1}/(2\beta-1)$; $\beta\neq\frac{1}{2}$ *and if* $\beta=\frac{1}{2}$ *then* $g(z)=-\log(1-z)$.

If $\beta\neq\frac{1}{2}$ then by direct computation we find that

$$f(z)=\frac{(2\beta-1)c-c(1-z)^{2\beta-1}+(2\beta-1)z(1-z)^{2\beta-2}}{(1+c)(2\beta-1)},$$

$$f'(z)=\frac{(1-z)^{2\beta-3}}{1+c}[(1+c)+(1-c-2\beta)z],$$

and, therefore

$$(1-\beta)+\frac{zf''(z)}{f'(z)}=\frac{(1-\beta)[(1+c)+2(2-\beta)z-(c+2\beta-1)z^2]}{(1-z)[(1+c)-(c+2\beta-1)z]}.$$

Thus the expression $(1-\beta)+zf''(z)/f'(z)$ vanishes for $z=-r_0$, hence $f(z)$ is not convex of order β in any disc $|z|<r$, $r>r_0$. Similarly for $\beta=\frac{1}{2}$ and $g(z)=-\log(1-z)$, sharpness of the theorem can be established.

If we take $c=1$ and $0\leqq\beta\leqq\frac{1}{2}$ in Theorem 2, then the following theorem of Padmanabhan [7] is obtained as a corollary to Theorem 2.

THEOREM [PADMANABHAN]. *Let* $g(z)\in C_\beta$. *Then* $f(z)=\frac{1}{2}[zg(z)]'\in C_\beta$ *for*

$$|z|<\left[\frac{(\beta-2)+(\beta^2+4)^{1/2}}{2\beta}\right].$$

THEOREM 3. *Let* $f(z)=(z^{1-c}/(1+c))[z^cF(z)]'$ *and* $g(z)=(z^{1-c}/(1+c))\times [z^cG(z)]'$, $c=1, 2, 3, \cdots$, $G(z)\in S_\beta^*$ *and* $F(z)\in\Gamma(\lambda,\beta)$ *with respect to* $G(z)$. *Then* $f(z)\in\Gamma(\lambda,\beta)$ *with respect to the function* $g(z)\in S_\beta^*$ *for* $|z|<r_0$. *The result is sharp.*

PROOF. Since $G(z)\in S_\beta^*$, from Theorem 1, we have

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\}\geqq\beta \quad \text{for } |z|<r_0.$$

Also, since $F(z)\in\Gamma(\lambda,\beta)$ with respect to $G(z)$, there exists an analytic function $\omega(z)$ satisfying the conditions of Schwarz's lemma such that

$$P(z)\equiv\frac{zF'(z)}{G(z)}=\frac{1-(1-2\lambda)\omega(z)}{1+\omega(z)}.$$

But $P(z)$ can also be written as

$$P(z)\int_0^z t^{c-1}g(t)\,dt = z^c f(z) - c\int_0^z t^{c-1}f(t)\,dt. \tag{2.10}$$

By differentiating (2.10) with respect to z, we obtain

$$\frac{zf'(z)}{g(z)} = P(z) + \frac{P'(z)}{g(z)}\,z^{1-c}\int_0^z t^{c-1}g(t)\,dt. \tag{2.11}$$

But

$$\frac{1}{z^c g(z)}\int_0^z t^{c-1}g(t)\,dt = \frac{G(z)}{cG(z) + zG'(z)}. \tag{2.12}$$

Further, since $G(z)\in S^*_\beta$, there exists an analytic function $V(z)$ satisfying the conditions of Schwarz's lemma, such that

$$\frac{zG'(z)}{G(z)} = \frac{1-(1-2\beta)V(z)}{1+V(z)}. \tag{2.13}$$

From (2.13) we have

$$\left[c + \frac{zG'(z)}{G(z)}\right]^{-1} = \left[\frac{(c+1)+(c-1+2\beta)V(z)}{1+V(z)}\right]^{-1}. \tag{2.14}$$

Hence we have from (2.12) and (2.14) that

$$\left[\int_0^z t^{c-1}g(t)\,dt\right]\Big/ z^c g(z) = \left[\frac{(c+1)+(c-1+2\beta)V(z)}{1+V(z)}\right]^{-1}. \tag{2.15}$$

Thus by using (2.15) we obtain from (2.11) that

$$\begin{aligned}\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} - \lambda &\geqq \operatorname{Re}\{P(z)-\lambda\} \\ &\quad\times\left[\frac{(c+1)-2(2-\beta)\,|z| - (c+2\beta-1)\,|z|^2}{(1-|z|)[(c+1)+(c+2\beta-1)\,|z|]}\right].\end{aligned} \tag{2.16}$$

The inequality (2.16) implies that $f(z)\in\Gamma(\lambda,\beta)$ with respect to the function $g(z)$ if $|z|<r_0$. This completes the proof of Theorem 3. Sharpness of the theorem follows from Theorem 2.

As a corollary to Theorem 3, Theorem H of Bernardi follows by taking $\beta=0$.

THEOREM 4. *Let* $F(z)=z+\sum_{n=2}^{\infty} a_n z^n$ *be regular and have the property* $\operatorname{Re}\{F'(z)\}>\beta$ *for* $|z|<1$, $f(z)=(1/(1+c))z^{1-c}[z^cF(z)]'$, $c=1,2,3,\cdots$. *Then* $\operatorname{Re}\{f'(z)\}>\beta$ *for* $|z|<r_1=[-1+(2+2c+c^2)^{1/2}/1+c]$. *The result is sharp.*

PROOF. The proof given by Bernardi for theorem [2, p. 317] remains valid except for the following change.

$$(1 + c)\mathrm{Re}\{f'(z) - \beta\} \geqq \mathrm{Re}\{P(z) - \beta\}\left[(1 + c) - \frac{2\,|z|}{1 - |z|^2}\right].$$

This result is sharp as is seen by the example,

$$F(z) = (2\beta - 1)z - 2(1 - \beta)\log(1 - z).$$

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