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STRONGLY DISSIPATIVE OPERATORS AND NONLINEAR EQUATIONS IN A FRÉCHET SPACE

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ABSTRACT. Suppose that X is a Fréchet space, Y is a Banach subspace of X, and A is a function from Y into X. Sufficient conditions are determined to insure that the equation $Ax=y$ ($y \in Y$) has a unique solution x_y which depends continuously on y. The techniques of this paper use the theory of dissipative operators in a Banach space, and the results are associated with the idea of admissibility of the space Y. Also, the equation $Ax=Cx+y$ is considered where C is completely continuous.

Let X be a Fréchet space over the real or complex field (i.e., X is a locally convex, complete, metrizable topological vector space—see e.g., [**10**, p. 85]). In this paper we assume the following:

(X1) $(q_n)_1^\infty$ is an increasing family of continuous seminorms on X which defines the topology of X (i.e., $q_n \leqq q_{n+1}$ and $X-\lim_{k\to\infty} x_k = x$ if and only if $\lim_{k\to\infty} q_n(x_k-x)=0$ for each n).

(X2) $Y=\{x\in X: \sup_n\{q_n(x)\}<\infty\}$ and $|x|=\sup_n\{q_n(x)\}$ for each x in Y.

Note that a sequence of seminorms satisfying (X1) always exists, and the space Y in (X2) is a complete normed space with norm $|\cdot|$. However, the members of Y depend on how the sequence $(q_n)_1^\infty$ is chosen.

In this paper, some results on strongly dissipative operators in a Banach space are used to establish analogous results for a class of operators which map Y into X. Recently, some fixed point theorems for completely continuous perturbations of Lipschitz continuous functions in locally convex spaces have been obtained by Cain and Nashed [**1**]. In this paper, a class of functions A from Y into X is considered which can be "approximated" by functions which are defined on X. Sufficient conditions are established to insure that the equation $Ax=y$ has a unique solution x_y for each y in Y, and the function B defined by $By=x_y$ for each y in Y has certain continuity properties. This result is closely associated with the notion of admissibility introduced by Massera and Schäffer [**6**]. Recently, admissibility has been used in studying existence and stability of solutions to Volterra integral equations—see Corduneanu [**2**], [**3**] and Miller [**8**].

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We also consider the solvability of the equation $Ax=Cx+y$ where C is a function from Y into Y which has certain compact properties and growth properties. Finally, two examples of integral equations on the half-line are given to illustrate when these techniques may be applied.

DEFINITION 1. If A is a function from X into X, then A is said to be compatible with q_n if whenever x is in X and $(x_k)_1^\infty$ is a sequence in X such that $\lim_{k\to\infty} q_n(x_k-x)=0$, it follows that $\lim_{k\to\infty} q_n(Ax_k-Ax)=0$.

For each positive integer n let $M_n=\{x\in X:q_n(x)=0\}$ and let $X/M_n=\{\phi(x):x\in X \text{ and } \phi(x)=x+M_n\}$. If $q_n^*(\phi(x))=q_n(x)$ for each $\phi(x)$ in X/M_n, then q_n^* is well defined and is a norm on the quotient space X/M_n. If A is a function from X into X which is compatible with q_n, then define $A^*\phi(x)=\phi(Ax)$ for each $\phi(x)$ in X/M_n. It is easy to see that A^* is well defined and continuous on X/M_n.

DEFINITION 2. Suppose that X/M_n, A and A^* are as in the above paragraph and let E_n denote the completion of the normed space X/M_n with the norm on E_n denoted by q_n'. Then A is said to be strongly dissipative with respect to q_n if there is a continuous function α_n from $[0,\infty)$ into $[0,\infty)$ and a continuous function A' from E_n into E_n such that $\alpha_n(0)=0$, $\alpha_n(s)>0$ if $s>0$, $A^*\phi(x)=A'\phi(x)$ if $\phi(x)$ is in X/M_n, and

$$\lim_{h\to 0+} [q_n'(x-y+h(A'x-A'y))-q_n'(x-y)]/h \leqq -\alpha_n(q_n'(x-y))$$

for all x and y in E_n.

REMARK. In many cases the quotient spaces X/M_n are complete—for example, the spaces considered in Examples 1 and 2 below have this property. If X/M_n is complete and A is compatible with q_n, note that A is strongly dissipative with respect to q_n only in case

$$\lim_{h\to 0+} [q_n(x-y+h(Ax-Ay))-q_n(x-y)]/h \leqq -\alpha_n(q_n(x-y))$$

for all x and y in X. Also, if the function A^* is uniformly continuous on bounded subsets of X/M_n, then A^* can be extended to a continuous function A' on E_n, and it can be shown that A is strongly dissipative with respect to q_n.

LEMMA 1. *Suppose that A is a function from X into X which is strongly dissipative with respect to q_n. Then, for each z in X and each $\varepsilon>0$, there is a x_z^ε in X such that $q_n(Ax_z^\varepsilon-z)\leqq\varepsilon$.*

INDICATION OF PROOF. If the function A' is as in Definition 2, it is easy to see that A' satisfies each of the suppositions of Theorem 2 in [7], and so there is a unique y' in E_n such that $q_n'(A'y'-\phi(z))=0$. Since X/M_n is dense in E_n and A' is continuous, there is an x_z^ε in X such that

$q'_n(A'\phi(x^\varepsilon_z)-\phi(z))\leqq\varepsilon$. Thus, $q_n(Ax^\varepsilon_x-z)=q^*_n(A^*\phi(x^\varepsilon_z)-\phi(z))\leqq\varepsilon$ and the lemma is true.

We now prove our main result.

THEOREM 1. *Suppose that conditions* (X1)—(X2) *are fulfilled*, α *is a continuous, increasing function from* $[0, \infty)$ *into* $[0, \infty)$ *such that* $\alpha(0)=0$ *and* $\lim_{s\to\infty}\alpha(s)=\infty$, A *is a function from* Y *into* X, *and* $(A_n)_1^\infty$ *is a sequence of functions from* X *into* X *such that*

(i) A_n *is strongly dissipative with respect to* q_n *for each* n;

(ii) $\lim_{h\to 0+}[q_n(x-y+h(A_nx-A_ny))-q_n(x-y)]/h\leqq-\alpha(q_n(x-y))$ *for each* x *and* y *in* X *and each* n;

(iii) *there is a number* $L\geqq 0$ *such that* $q_n(A_n0)\leqq L$ *for each* n;

(iv) *for each pair of positive numbers* K *and* δ *there is a positive integer* $N(K, \delta)$ *such that if* $j\geqq i\geqq N(K, \delta)$ *and* x *is a member of* X *such that* $q_j(x)\leqq K$, *then* $q_i(A_ix-A_jx)\leqq\delta$; *and*

(v) *if* K *is a positive number, then* $\lim_{i\to\infty}q_i(A_ix-Ax)=0$, *uniformly for* x *in* Y *with* $|x|\leqq K$.

Then, for each z *in* Y, *there is a unique* x_z *in* Y *such that* $Ax_z=z$. *Furthermore, if the function* B *from* Y *into* Y *is defined by* $Bz=x_z$ *for each* z *in* Y, *then* $|Bz-Bw|\leqq\alpha^{-1}(|z-w|)$ *for all* z *and* w *in* Y.

REMARK. Note that the function A in Theorem 1 need not be defined on all of X and does not necessarily map Y into Y. Furthermore, by supposition (ii) if the quotient spaces X/M_n are complete for each n, then we need only assume that A_n is compatible with q_n in supposition (i)—see remark following Definition 2.

PROOF OF THEOREM 1. Note that supposition (ii) implies that

$$q_n(x-y)\leqq\alpha^{-1}(q_n(A_nx-A_ny)) \tag{1}$$

for all x and y in X. By Lemma 1 and suppositions (i) and (ii), there is an x^n_z in X such that $q_n(A_nx^n_z-z)\leqq n^{-1}$ for each z in Y and positive integer n. Since α^{-1} is increasing, we have from (1) and supposition (iii) that

$$q_n(x^n_z)\leqq\alpha^{-1}(q_n(A_nx^n_z-A_n0))\leqq\alpha^{-1}(|z|+1+L) \tag{2}$$

for each n. Let ε be a positive number and let $\delta>0$ be such that $\alpha^{-1}(s)\leqq\varepsilon$ whenever $0\leqq s\leqq 2\delta$. If N is as in (iv) and $j\geqq i\geqq N(\alpha^{-1}(|z|+1+L), \delta)$, then $q_j(x^j_z)\leqq\alpha^{-1}(|z|+1+L)$, so

$$\begin{aligned} q_i(A_ix^i_z-A_ix^j_z) &\leqq q_i(z-A_ix^j_z)+i^{-1} \\ &\leqq q_i(z-A_jx^j_z)+q_i(A_jx^j_z-A_ix^j_z)+i^{-1} \\ &\leqq 2i^{-1}+q_i(A_jx^j_z-A_ix^j_z)\leqq 2i^{-1}+\delta. \end{aligned} \tag{3}$$

Now let m be a positive integer such that

$$m \geqq \max\{N(\alpha^{-1}(|z|+1+L), \delta), 2/\delta\}.$$

If $j \geqq i \geqq m$, then by (1), (3), and the choice of δ,

$$q_i(x_z^i - x_z^j) \leqq \alpha^{-1}(q_i(A_i x_z^i - A_i x_z^j)) \leqq \alpha^{-1}(2\delta) \leqq \varepsilon. \tag{4}$$

It now follows easily from (4) and (X1) that $(x_z^k)_1^\infty$ is a Cauchy sequence in X. Since X is complete, there is an x_z in X such that $X-\lim_{k\to\infty} x_z^k = x_z$. By (2),

$$q_n(x_z) = \lim_{k\to\infty} q_n(x_z^k) \leqq \limsup_{k\to\infty} q_k(x_z^k) \leqq \alpha^{-1}(|z| + 1 + L)$$

for each n, so x_z is in Y with $|x_z| \leqq \alpha^{-1}(|z|+1+L)$. Letting $j\to\infty$ in the term on the left side of (3) shows that $\lim_{i\to\infty} q_i(A_i x_z^i - A_i x_z) = 0$. Thus, from supposition (v), if n is a positive integer,

$$\begin{aligned} q_n(Ax_z - z) &\leqq \lim_{i\to\infty} q_i(Ax_z - z) = \lim_{i\to\infty} q_i(A_i x_z - z) \\ &\leqq \lim_{i\to\infty} q_i(A_i x_z - A_i x_z^i) + i^{-1} = 0. \end{aligned}$$

Hence $Ax_z = z$. If y is in Y and $Ay = z$, then by (1) and supposition (v),

$$\begin{aligned} q_n(y - x_z) &\leqq \lim_{i\to\infty} q_i(y - x_z) \leqq \lim_{i\to\infty} \alpha^{-1}(q_i(A_i y - A_i x_z)) \\ &\leqq \lim_{i\to\infty} \alpha^{-1}(q_i(A_i y - z) + q_i(z - A_i x_z)) = \alpha^{-1}(0) = 0 \end{aligned}$$

for each n. Consequently $y = x_z$ and the function B defined in the statement of the theorem is well defined. Furthermore, if z and w are in Y and $(x_z^i)_1^\infty$ and $(x_w^i)_1^\infty$ are as constructed above, then by (1) and the fact that $q_i(A_i x_z^i - A_i x_w^i) \leqq q_i(z-w) + 2i^{-1}$,

$$\begin{aligned} q_n(x_z - x_w) &= \lim_{i\to\infty} q_n(x_z^i - x_w^i) \leqq \limsup_{i\to\infty} q_i(x_z^i - x_w^i) \\ &\leqq \limsup_{i\to\infty} \alpha^{-1}(q_i(A_i x_z^i - A_i x_w^i)) \\ &\leqq \limsup_{i\to\infty} \alpha^{-1}(q_i(z - w) + 2i^{-1}) \leqq \alpha^{-1}(|z - w|). \end{aligned}$$

Thus,

$$|Bz - Bw| = \sup\{q_n(x_z - x_w) : n = 1, 2, \cdots\} \leqq \alpha^{-1}(|z - w|)$$

and the proof of Theorem 1 is complete.

LEMMA 2. *Let the suppositions of Theorem* 1 *be fulfilled and for each* $R>0$ *let* $Q_R = \{x \in Y : |x| \leqq R\}$. *Then, considering* Q_R *with the topology induced by* X, *the function* B *defined in Theorem* 1 *is continuous from* Q_R *into* X.

INDICATION OF PROOF. Let x be in Q_R and let $(x_k)_1^\infty$ be a sequence in Q_R such that $\lim_{k\to\infty} q_n(x_k-x)=0$ for each n. Let n be a positive integer, let $\varepsilon>0$, and let $\delta>0$ be such that $\alpha^{-1}(s)\leqq\varepsilon$ whenever $0\leqq s\leqq\delta$. Note that if y is in Q_R, then $|By|\leqq|By-BO|+|BO|\leqq\alpha^{-1}(R)+|BO|$. By supposition (v) of Theorem 1, let the integer $m\geqq n$ be such that $q_m(A_my-Ay)\leqq\delta/3$ for all y in Y with $|y|\leqq\alpha^{-1}(R)+|BO|$. Let p be a positive integer such that $q_m(x_k-x)\leqq\delta/3$ whenever $k\geqq p$. If $k\geqq p$, we have from (1), the choice of δ, and the fact that α^{-1} is increasing, that

$$\begin{aligned} q_n(Bx_k - Bx) &\leqq q_m(Bx_k - Bx) \leqq \alpha^{-1}(q_m(A_mBx_k - A_mBx)) \\ &\leqq \alpha^{-1}(q_m(A_mBx_k - ABx_k) + q_m(ABx_k - ABx) \\ &\qquad\qquad + q_m(ABx - A_mBx)) \\ &\leqq \alpha^{-1}(\delta/3 + q_m(x_k - x) + \delta/3) \leqq \alpha^{-1}(\delta) \leqq \varepsilon. \end{aligned}$$

Thus $\lim_{k\to\infty} q_n(Bx_k-Bx)=0$ for each n, and the assertion of the lemma follows.

THEOREM 2. *Let the suppositions of Theorem* 1 *be fulfilled and suppose that C is a function from Y into Y such that*

$$\limsup_{|x|\to\infty} \alpha^{-1}(|Cx|)/|x| = \beta < 1.$$

Suppose further that at least one of the following is satisfied:

(i) *C is continuous from the Banach space Y into itself and maps bounded subsets of Y into relatively compact subsets of Y*; *or*

(ii) *if Q_R is as in Lemma* 2 (*with the topology induced by X*), *then, for each $R>0$, C is continuous from Q_R into X and the image of Q_R under C is relatively compact in X.*

Then, for each z in Y, there is a y_z in Y such that $Ay_z-Cy_z=z$.

INDICATION OF PROOF. Note that we need only show that there is an x_0 in Y such that $Ax_0-Cx_0=0$, or equivalently, $B\cdot Cx_0=x_0$ where B is as defined in Theorem 1. Since $\limsup_{|x|\to\infty} |B\cdot Cx|/|x|<1$, let $r_1>0$ be such that $|B\cdot Cx|\leqq|x|$ whenever $|x|\geqq r_1$ and let $r_2=\sup\{|B\cdot Cx|:|x|\leqq r_1\}$. If $R=\max\{r_1, r_2\}$, then $B\cdot C$ maps Q_R into Q_R. If (i) holds the theorem follows from the Schauder fixed point theorem, and if (ii) holds the theorem follows from Lemma 2 and the Tychonov fixed point theorem (see e.g. [**5**, Theorem 5, p. 456]).

EXAMPLE 1. Let X be the space $LL^2[0, \infty)$ of all measurable functions x from $[0, \infty)$ into the real numbers such that $q_n(x)=(\int_0^n |x(s)|^2\, ds)^{1/2}<\infty$ for each positive integer n. Note that Y is the space $L^2[0, \infty)$ and $|x|=(\int_0^\infty |x(s)|^2\, ds)^{1/2}$ for each x in Y. Let a and b be symmetric, measurable, locally integrable functions defined on $[0, \infty)^2$ such that the operator

$Tx(t)=\int_0^t a(t, s)x(s)\,ds$ maps $LL^2[0, \infty)$ continuously into $LL^2[0, \infty)$ and the operator $Sx(t)=\int_0^\infty b(t, s)x(s)$ maps $L^2[0, \infty)$ into $L^2(0, \infty)$. Assume further that (a) $\int_0^n\int_0^n a(t, s)x(s)x(t)\,ds\,dt \geqq 0$ for each x in $LL^2[0, \infty)$ and each n; (b) $\int_0^n\int_0^n b(t, s)x(s)x(t)\,ds\,dt \geqq 0$ for each x in $LL^2[0, \infty)$ and each n; and (c) $\int_0^\infty\int_0^\infty |b(t, s)|^2\,ds\,dt < \infty$. Define $Ax=-x-Tx-Sx$ for each x in $L^2[0, \infty)$. Then the suppositions of Theorem 1 are fulfilled with $\alpha(s)=-s$ for all $s \geqq 0$ and $A_n x=-x-Tx-S_n x$ where $S_n x(t)=\int_0^n b(t, s)x(s)\,ds$ for each x in $LL^2[0, \infty)$. Suppositions (i) and (iii) of Theorem 1 are easily seen to be true. Supposition (ii) is immediate from (a) and (b) above and the fact that

$$2\int_0^n\left[\int_0^t a(t, s)x(s)\,ds\right]x(t)\,dt = \int_0^n\int_0^n a(t, s)x(s)x(t)\,ds\,dt,$$

which follows from the symmetry of a. Since condition (c) above implies $\lim_{p\to\infty}\int_0^p\int_p^\infty |b(t, s)|^2\,ds\,dt=0$, it is easy to see that suppositions (iv) and (v) are fulfilled. Thus, for each z in $L^2[0, \infty)$, there is a unique x_z in $L^2[0, \infty)$ such that

$$x_z(t) + \int_0^t a(t, s)x_z(s)\,ds + \int_0^\infty b(t, s)x_z(s)\,ds = z(t) \tag{5}$$

for almost all t in $[0, \infty)$. Also, $\int_0^\infty |x_z(s)|^2\,ds \leqq \int_0^\infty |z(s)|^2\,ds$. Note the operator defined by the left side of (5) is not necessarily defined on all of $LL^2(0, \infty)$ and need not map $L^2[0, \infty)$ into itself.

Example 2. Let $C_c[0, \infty)$ denote the Fréchet space of all continuous functions x from $[0, \infty)$ into the m dimensional space R^m (with $\|\cdot\|$ denoting an appropriate norm on R^m) with the topology generated by uniform convergence on compact subsets of $[0, \infty)$. Define $q_n(x)=\max\{\|x(t)\| : 0 \leqq t \leqq n\}$ for each x in $C_c[0, \infty)$ and each positive integer n. Then Y is the space $BC[0, \infty)$ of all x in $C_c[0, \infty)$ such that $|x|=\sup\{\|x(t)\| : t \geqq 0\} < \infty$. Now let f and g be continuous functions from $[0, \infty)^2 \times R^m$ into R^m such that (a) $f(t, s, 0)=g(t, s, 0)=0$ for all (t, s) in $[0, \infty)^2$; (b) $f(t, s, \xi)=0$ for all (t, s, ξ) in $[0, \infty)^2 \times R^m$ with $s>t$; and there are continuous functions θ and ϕ from $[0, \infty)^2$ into $[0, \infty)$ such that (c) $\sup\{\int_0^t \theta(t, s)\,ds : t \geqq 0\}=\lambda<1$; (d) $\sup\{\int_0^\infty \phi(t, s)\,ds : t \geqq 0\}=\gamma<1-\lambda$; (e) $\|f(t, s, \xi_1)-f(t, s, \xi_2)\| \leqq \theta(t, s)\|\xi_1-\xi_2\|$ for all (t, s, ξ_1) and (t, s, ξ_2) in $[0, \infty)^2 \times R^m$; and (f) $\|g(t, s, \xi)\| \leqq \phi(t, s)\,\|\xi\|$ for all (t, s, ξ) in $[0, \infty)^2 \times R^m$. If S is the integral operator defined on $C_c[0, \infty)$ by $Sx(t)=\int_0^t f(t, s, x(s))\,ds$, then the suppositions of Theorem 1 are easily seen to be fulfilled with $A=-x+Sx$ for each x in $BC[0, \infty)$, $A_n x=-x+Sx$ for each x in $C_c[0, \infty)$, and $\alpha(s)=(1-\lambda)s$ for each $s \geqq 0$. Let T be the integral operator defined in $BC[0, \infty)$ by $Tx(t)=\int_0^\infty g(t, s, x(s))\,ds$ and suppose that T maps $BC[0, \infty)$ into $BC[0, \infty)$. Note that conditions (d) and (f) above imply that $|Tx| \leqq \gamma |x|$ for all x in $BC[0, \infty)$. Now let z be in $BC[0, \infty)$ and let $Cx = Tx + z$ for each x in $BC[0, \infty)$. If B is as in Theorem 1 (i.e., $A \cdot Bx = x$ for all x in $BC[0, \infty)$), then $BO = 0$ and we have that $|Bx| \leqq (1-\lambda)^{-1}|x|$ for all x in $BC[0, \infty)$. It now follows easily that the operator $B \cdot C$ maps the ball $Q = \{x \in BC[0, \infty) : |x| \leqq |z|(1-\gamma-\lambda)^{-1}\}$ into itself. Thus, if C is completely continuous for the $BC[0, \infty)$ topology on Q or if C is completely continuous for the $C_c[0, \infty)$ topology on Q (note that this is the case if

$$\lim_{p \to \infty} \left[\sup \left\{ \int_p^\infty \phi(t, s)\, ds : 0 \leqq t \leqq p \right\} \right] = 0,$$

where ϕ is as in (d)), then there is an x_z in $BC[0, \infty)$ with

$$|x_z| \leqq |z|\,(1 - \gamma - \lambda)^{-1}$$

such that

$$x_z(t) - \int_0^t f(t, s, x_z(s))\, ds = \int_0^\infty g(t, s, x_z(s))\, ds + z(t) \tag{6}$$

for all t in $[0, \infty)$. Furthermore, since $|B \cdot Cx| < |x|$ if $|x| > |z|(1-\gamma-\lambda)^{-1}$, all solutions x_z to (6) satisfy $|x_z| \leqq |z|(1-\gamma-\lambda)^{-1}$. In particular $|x_z| \to 0$ as $|z| \to 0$, so we have a type of stability criteria for the zero solution of equation (6) when $z(t) = 0$ for all $t \geqq 0$.

REMARK. In Example 2, we need only assume that the inequalities (e) and (f) hold in some neighborhood of the origin in R^m. This follows from the fact that if $r > 0$ and h is a function from $Q_r = \{\xi \in R^m : \|\xi\| \leqq r\}$ into R^m such that $\|h(\xi_1) - h(\xi_2)\| \leqq k\|\xi_1 - \xi_2\|$ for all ξ_1 and ξ_2 in Q_r, then there is a function h^* from R^m into R^m such that $h^*(\xi) = h(\xi)$ for all ξ in Q_r and $\|h^*(\xi_1) - h^*(\xi_2)\| \leqq k^*\|\xi_1 - \xi_2\|$ for all ξ_1 and ξ_2 in R^m where $k \leqq k^* \leqq 2k$ (take $h^*(\xi) = h(r\xi/\|\xi\|)$ if $\|\xi\| > r$). Note that $k^* = k$ if $\|\cdot\|$ is the Euclidean norm on R^m. Thus the stability and existence criteria established in Example 2 give some improvements to those of Miller, Nohel and Wong [**9**].

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