PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 1, March 1972

ON OPERATORS WITH RATIONAL RESOLVENT

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ABSTRACT. It is shown that a bounded linear operator T on a complex Banach space into itself has a rational resolvent if and only if every bounded linear operator which commutes with every bounded linear operator that commutes with T can be expressed as a polynomial in T.

In this note, we present a characterization of the Caradus class $\mathfrak{F}$ of bounded linear operators on a complex Banach space $\mathfrak{X}$ into itself [**1**], that is the class of bounded linear operators with rational resolvent [**5**, p. 314]. This confirms a suggestion of Caradus in [**2**].

We shall appeal to a version of the minimal equation theorem in which the following terminology is used.

DEFINITION. *Let T be a bounded linear operator on a complex Banach space $\mathfrak{X}$ into itself. We define the* **spectral order** *$\nu_T(\lambda)$ of the point λ of the spectrum of T to be the order of λ as a pole of the resolvent of T; if λ is not a pole of the resolvent of T, we give $\nu_T(\lambda)$ the conventional value ∞.*

We could extend this definition by putting $\nu_T(\lambda)=0$ when λ is in the resolvent set of T.

THEOREM 1 (MINIMAL EQUATION THEOREM). *Let T be a bounded linear operator on a complex Banach space $\mathfrak{X}$ into itself, and let f and g be functions holomorphic on open sets which contain the spectrum of T. Then $f(T)=g(T)$ if and only if, for every point λ of the spectrum of T, $f^{(r)}(\lambda)=g^{(r)}(\lambda)$ when r is an integer with $0\leqq r<\nu_T(\lambda)$, where $\nu_T(\lambda)$ is the spectral order of λ.*

This follows immediately from [**3**, Theorem VII.3.16, p. 571].

THEOREM 2. *Let $\mathfrak{X}$ be a complex Banach space. The bounded linear operator T on $\mathfrak{X}$ into itself belongs to the Caradus class $\mathfrak{F}$ if and only if it has the property that a bounded linear operator S (on $\mathfrak{X}$ into itself) commutes with every bounded linear operator which commutes with T only if there is a polynomial p such that $S=p(T)$.*

Received by the editors July 6, 1971.

AMS 1970 *subject classifications*. Primary 47A10, 47B10; Secondary 46B99, 47A60.

Key words and phrases. Bounded linear operator, rational resolvent, Caradus class, pole of the resolvent, spectrum, minimal equation theorem.

The necessity of this condition was proved in [**4**]. To prove that it is sufficient, suppose it satisfied. We put $S=\exp T$; this certainly commutes with every bounded linear operator which commutes with T. By hypothesis there is a polynomial p such that $S=p(T)$. Since almost all derivatives of a polynomial function vanish everywhere, and no derivative of the exponential function vanishes anywhere, it follows from Theorem 1 that $\nu_T(\lambda)$ is finite (in fact at most $n+1$ where n is the degree of p) for every point λ of the spectrum of T. Thus every point of the spectrum of T is a pole of the resolvent of T. But the spectrum of T is compact, and a limit point of poles is not a pole. We conclude that the spectrum of T consists of a finite number of poles of the resolvent of T, that is $T\in\mathfrak{F}$.

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