PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 2, April 1972

A NOTE ON $\mathscr{Z}$-REALCOMPACTIFICATIONS

ANTHONY J. D'ARISTOTLE

ABSTRACT. Orrin Frink showed that the real-valued functions over a Tychonoff space X which may be continuously extended to $\omega(\mathscr{Z})$, the Wallman-type compactification associated with a normal base $\mathscr{Z}$ for X, are those which are *$\mathscr{Z}$-uniformly continuous.*

Let $\mathscr{Z}$ be a delta normal base on a Tychonoff space X, and let $\eta(\mathscr{Z})$ be the corresponding $\mathscr{Z}$-realcompactification of X. In this note we show that *countable $\mathscr{Z}$-uniform continuity* is a sufficient but not a necessary condition for continuously extending real-valued functions over X to $\eta(\mathscr{Z})$.

In [3], Orrin Frink utilized the notion of a *normal base* to obtain Hausdorff compactifications for Tychonoff or completely regular T_1 spaces X. A normal base $\mathscr{Z}$ for the closed sets of a space X is a base which is a disjunctive ring of sets, disjoint members of which may be separated by disjoint complements of members of $\mathscr{Z}$. Frink proved that if $\mathscr{Z}$ is a normal base for a T_1 space X, then the space $\omega(\mathscr{Z})$ consisting of the $\mathscr{Z}$-ultrafilters, is a Hausdorff compactification of X. By choosing different normal bases $\mathscr{Z}$ for a noncompact space X, different Hausdorff compactifications of X may be obtained.

In [1], Alo and Shapiro used $\mathscr{Z}$-ultrafilters from a *delta normal base* (a normal base closed under countable intersections) to introduce a new space $\eta(\mathscr{Z})$ consisting of those $\mathscr{Z}$-ultrafilters with the *countable intersection property*. To each delta normal base $\mathscr{Z}$ on X there corresponds a delta normal base $\mathscr{Z}^*$ on $\eta(\mathscr{Z})$, and they have shown that every $\mathscr{Z}^*$-ultrafilter with the countable intersection property is fixed, i.e., $\eta(\mathscr{Z})$ is $\mathscr{Z}^*$-realcompact. For many delta normal bases $\mathscr{Z}$, $\eta(\mathscr{Z})$ is realcompact in the usual sense but, in [5], it has been shown that this is not always the case.

A real function f defined over a space X with normal base $\mathscr{Z}$ is said to be *$\mathscr{Z}$-uniformly continuous* if for every positive epsilon there exists a finite open cover of X by $\mathscr{Z}$-complements, on each of which the oscillation of f is less than epsilon. Frink showed that f may be continuously extended to $\omega(\mathscr{Z})$ if and only if f is $\mathscr{Z}$-uniformly continuous.

Presented to the Society, January 22, 1971; received by the editors May 21, 1971.

AMS 1970 *subject classifications*. Primary 54C20, 54C30, 54C45.

Key words and phrases. Normal base, delta normal base, $\mathscr{Z}$-uniform continuity, countable $\mathscr{Z}$-uniform continuity.

The analogous condition for a space X with a delta normal base is *countable $\mathscr{Z}$-uniform continuity*: A real function f defined over a space X with delta normal base $\mathscr{Z}$ is countable $\mathscr{Z}$-uniformly continuous if corresponding to every positive epsilon there exists a finite or denumerable open cover of X by $\mathscr{Z}$-complements, on each of which the oscillation of f is less than epsilon. In this note we show that countable $\mathscr{Z}$-uniform continuity is a sufficient but not a necessary condition for extendibility to $\eta(\mathscr{Z})$.

For definitions and a thorough discussion of the results cited above, the reader is referred to **[1]** and **[3]**.

THEOREM. *Every countable $\mathscr{Z}$-uniformly continuous function on X can be continuously extended to a real-valued function on $\eta(\mathscr{Z})$.*

PROOF. If $\mathscr{Z}$ is a delta normal base for X, let $\mathscr{U}$ be the collection of all free $\mathscr{Z}$-ultrafilters on X with the countable intersection property. Then $\eta(\mathscr{Z})=X\cup\mathscr{U}$ and the topology for $\eta(\mathscr{Z})$ is that obtained by taking as a base for the closed sets the family of all sets A^* of the form $A\cup\{\mathscr{A}\in\mathscr{U}|A\in\mathscr{A}\}$ where $A\in\mathscr{Z}$.

If f is a countable $\mathscr{Z}$-uniformly continuous function on X, we define a function g which extends f from X to $\eta(\mathscr{Z})$ as follows. If $x\in X$ we let $g(x)=f(x)$. If $\mathscr{A}\in\mathscr{U}$ then the family $S_{\mathscr{A}}=\{f(A):A\in\mathscr{A}\}$ has the finite intersection property and is therefore a subbase for the filter $\mathscr{F}_{\mathscr{A}}$ consisting of all supersets of finite intersections of members of $S_{\mathscr{A}}$. The filter $\mathscr{F}_{\mathscr{A}}$ is a Cauchy filter and therefore converges uniquely to a real number which we call $g(\mathscr{A})$.

That g is continuous at each point of X is readily verified. It remains to show that g is continuous at each point $\mathscr{B}\in\mathscr{U}$. Let the family $\{X-C_j\}_{j=1}^{\infty}$ be a denumerable cover of X by $\mathscr{Z}$-complements, on each member of which the oscillation of f is less than $\varepsilon/3$ (we lose no generality in assuming that the cover of X is denumerable). We may suppose that $C_1\notin\mathscr{B}$ so that there is an element $Q\in\mathscr{B}$ with $Q\subseteq X-C_1$. We show that

$$g[\eta(\mathscr{Z})-C_1^*]$$
$$=g[(X-C_1)\cup\{\mathscr{A}\in\mathscr{U}:\exists P\in\mathscr{A}\text{ with }P\subseteq X-C_1\}]\subseteq S(g(\mathscr{B}),\varepsilon).$$

Now $g(\mathscr{B})\in\mathrm{cl}_R f(Q)$ and we choose $q\in Q$ so that $|g(\mathscr{B})-f(q)|<\varepsilon/3$. If $y\in X-C_1$ we then have

$$|g(\mathscr{B})-g(y)|\leqq|g(\mathscr{B})-f(q)|+|f(q)-g(y)|<\varepsilon/3+\varepsilon/3<\varepsilon.$$

It therefore follows that $g(X-C_1)\subseteq S(g(\mathscr{B}),\varepsilon)$. If $\mathscr{A}\in\mathscr{U}$ and there is a $P\in\mathscr{A}$ with $D\subseteq X-C_1$, we choose a point $p\in P$ satisfying $|g(\mathscr{A})-f(p)|<\varepsilon/3$.

The points q and p are members of $X-C_1$ and so

$$|g(\mathscr{A})-g(\mathscr{B})| \leqq |g(\mathscr{A})-f(p)|+|f(p)-f(q)|+|f(q)-g(\mathscr{B})|$$
$$< \varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon.$$

Thus g is a continuous, real-valued function on $\eta(\mathscr{Z})$.

In case $\mathscr{Z}$ is the collection of all zero-sets of a Tychonoff space X, then $\eta(\mathscr{Z})$ is precisely the Hewitt realcompactification of X. By observing that a real-valued function on a topological space is continuous if and only if it is countable zero-set uniformly continuous, we have the following well-known result [**4**] as a

COROLLARY. *Every continuous, real-valued function on a Tychonoff space X can be continuously extended to a real-valued function on X, the Hewitt realcompactification of X.*

However, countable $\mathscr{Z}$-uniform continuity is not a necessary condition for extendibility. To see this, we will make use of an example given by A. Steiner and E. Steiner in [**5**].

Let $X=[0, 1]$ with the discrete topology, let $\mathscr{F}_1$ be the family of all closed subsets of X with respect to the usual topology on $[0, 1]$, and let $\mathscr{F}_2$ be the family of all subsets of X which are finite or whose complement is countable. Then the family $\mathscr{Z}$ of countable intersections of finite unions of members of $\mathscr{F}_1 \cup \mathscr{F}_2$ is a delta normal base. Furthermore, if $\mathscr{A}$ is $\mathscr{Z}$-ultrafilter, then $\mathscr{A}$, being prime, contains a decreasing sequence of closed intervals whose lengths converge to 0; so if $\mathscr{A}$ has the countable intersection property, then $\mathscr{A}$ is fixed. Thus $\eta(\mathscr{Z})=X$.

The $\mathscr{Z}$-complements are all sets U of the form either:

(i) U is denumerable or finite;

(ii) U is open with respect to the usual topology on $[0, 1]$; or

(iii) $U=V_1 \cup V_2$ where V_1 has form (i) and V_2 form (ii). The function $f: X \rightarrow R$ equal to 1 on the rationals and 0 on the irrationals is certainly extendible to $\eta(\mathscr{Z})=X$. However, if $\{U_i\}_{i=1}^{n \text{ or } \infty}$ is a cover of X by $\mathscr{Z}$-complements, then at least one U_i has form (ii) or (iii). Hence f is not countable $\mathscr{Z}$-uniformly continuous.

A delta normal base $\mathscr{Z}$ is a strong delta normal base if each $A \in \mathscr{Z}$ is a countable intersection of $\mathscr{Z}$-complements. For such normal bases, $\eta(\mathscr{Z})$ is always realcompact [**2**].

The normal base $\mathscr{Z}$ in the above example is easily seen to be strong delta. Hence, also in this situation, countable $\mathscr{Z}$-uniform continuity is a sufficient but not a necessary condition for extendibility.

REFERENCES

1. R. A. Alo and H. L. Shapiro, *$\mathscr{Z}$-real compactifications and normal bases*, J. Austral. Math. Soc. **9** (1969), 489–495. MR **39** #3455.

2. ———, *Wallman compact and realcompact spaces*, Proc. Sympos. Contributions to Extension Theory of Topological Structures (Berlin, 1967), Academic Press, New York; Deutsch. Verlag Wissenschaften, Berlin, 1969, pp. 9–14. MR **40** #872.

3. O. Frink, *Compactifications and semi-normal spaces*, Amer. J. Math. **86** (1964), 602–607. MR **29** #4028.

4. L. Gillman and M. Jerison, *Rings of continuous functions*, The University Series in Higher Math., Van Nostrand, Princeton, N.J., 1960. MR **22** #6994.

5. A. K. Steiner and E. F. Steiner, *Nest generated intersection rings in Tychonoff spaces*, Trans. Amer. Math. Soc. **148** (1970), 589–601. MR **41** #7637.

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, COLLEGE AT GENESEO, GENESEO, NEW YORK 14454