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K_1 OF A COMMUTATIVE VON NEUMANN REGULAR RING

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ABSTRACT. Let A be a commutative regular ring (in the sense of von Neumann), and let q be an ideal in A. Then $K_1(A, q) = U(A, q)$.

A ring A is called regular (in the sense of von Neumann) if for all $a \in A$, there exists $s \in A$ such that $a = asa$. Some properties of these rings are discussed in [**2**, Chapter I, §2, Exercises 16, 17, and Chapter II, §4, Exercise 16]. We will consider only A commutative, with unit. Let q be an ideal in A. Then we show that $K_1(A, q) = U(A, q)$, where $U(A, q)$ denotes the units of A which are congruent to 1 mod q. The groups $K_1(A, q)$ are thus as simple as possible, as might be expected since A is zero dimensional. If A has a finite number of ideals, then A is the direct product of a finite number of fields, and these results are well known, for example, by Theorem 9.1, p. 266, of [**1**]. However, if A has an infinite number of ideals, then our results seem to be new. By [**1**, Chapter 5, §2], the determination of $K_1(A, q)$ is equivalent to the determination of all normal subgroups of $GL(A)$. The notation is as in Chapter 5 of [**1**]. The result where $q \neq A$ was suggested by the referee.

We first consider the case $q = A$, where we write $K_1(A, A) = K_1(A)$. There is a surjective homomorphism $K_1(A) \to U(A)$ and this can be split by mapping $U(A) \to GL_1(A)$. Write $K_1(A) = U(A) \oplus SK_1(A)$. In order to prove that $SK_1(A) = 0$ it is sufficient to prove (in view of the Whitehead Lemma, p. 226 of [**1**]) that if $a \in GL_n(A)$, then there exist b and c in $E_n(A)$ such that bac is a diagonal matrix.

First I will make an observation that is valid for any ring A. Suppose $A = A_1 \times A_2 \times \cdots \times A_m$. Then $GL_n(A) = GL_n(A_1) \times GL_n(A_2) \times \cdots \times GL_n(A_m)$, and under this isomorphism $E_n(A) = E_n(A_1) \times \cdots \times E_n(A_m)$. If $a \in GL_n(A)$, then a corresponds to $(a_1, a_2, \cdots, a_m)$, where $a = a_1 + a_2 + \cdots + a_m$, and the matrix a_i has coefficients in A_i.

Assume again that A is a commutative regular ring (in the sense of von Neumann). I will prove by induction on n that if $a \in GL_n(A)$, then there exist b, $c \in E_n(A)$ such that bac is diagonal.

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Let a_{11} be the entry in the upper left hand corner of a. Since $a_{11}A$ is generated by an idempotent, we have a direct product decomposition $A=A_1\times S_1$, such that a_{11} projects to a unit a_{11} in A_1, and to zero in S_1. Write $a=a_1+b_1$, with $a_1\in GL_n(A_1)$ and $b_1\in GL_n(S_1)$. Let b_{21} be the entry in the second row, first column of b_1. Write $S_1=A_2\times S_2$, where b_{21} projects to a unit in A_2 and to zero in S_2. We then have $a=a_1+a_2+b_2$, with $a_i\in GL_n(A_i)$, $i=1$, 2, and $b_2\in GL_n(S_2)$, and b_2 has the first two entries in the first column zero. Continue down the first column in this manner. Eventually we get a decomposition $A=A_1\times A_2\times\cdots\times A_n$, and $a=a_1+a_2+\cdots+a_n$, where $a_i\in GL_n(A_i)$ and a_i has a unit (in A_i) as the ith entry in the first column. (Some of the rings A_i might be trivial.) By using elementary row and column transformations over each A_i, we reduce a to the form where it has a unit in the upper left hand corner, and all other entries in the first row and column are zero. The proof is now completed by induction on n.

Now we consider the case where $q\neq A$. By definition, $K_1(A,q)=GL(A,q)/E(A,q)$, where $GL(A,q)=\ker(GL(A)\to GL(A/q))$ and $E(A,q)$ is the normal subgroup of $E(A)$ generated by the q-elementary matrices. The sequence $0\to E(A,q)\to E(A)\to E(A/q)\to 0$ is not exact for all A and q since $x\in\ker(E(A)\to E(A/q))$ need not lie in $E(A,q)$. It is, however, exact for the A considered here. First of all, it is exact if q is of finite type (hence idempotent generated) by Proposition 1.5, p. 451 of [1].

Suppose $x\in\ker(E(A)\to E(A/q))$. Then there is an ideal q' of finite type, with $q'\subset q$ and $x\in\ker(E(A)\to E(A/q'))$. Then $x\in E(A,q')\subset E(A,q)$. Hence we have an exact sequence $0\to E(A,q)\to E(A)\to E(A/q)\to 0$. It now follows from the serpent diagram (p. 17 of [2]) that $0\to K_1(A,q)\to K_1(A)\to K_1(A/q)$ is exact. Since A/q is also a von Neumann regular ring, we have that $K_1(A,q)=\ker(U(A)\to U(A/q))$ and this by definition equals $U(A,q)$.

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