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# EMBEDDING RATIONAL DIVISION ALGEBRAS[1]

BURTON FEIN

ABSTRACT. Necessary and sufficient conditions are given for two $K$-division rings, $K$ an algebraic number field, to have precisely the same set of subfields. Using this, an example is presented of two $K$-division rings having precisely the same set of subfields such that only one of the division rings can be embedded in a $Q$-division ring.

Let $K$ be a field. By a $K$-division ring we mean a finite-dimensional division algebra with center $K$. If $D$ is a $K$-division ring and $k$ is a field, $k \subset K$, we say that $D$ is $k$-adequate if $D$ can be embedded in a $k$-division ring. Similarly, if $L$ is a field, we say that $L$ is $k$-adequate if $L$ is a subfield of some $k$-division ring. Clearly, if $D$ is $k$-adequate then so is every subfield of $D$. In **[4]** the converse was raised: if every subfield of $D$ is $k$-adequate, must $D$ be $k$-adequate? We show that the answer to this question is no by exhibiting two $K$-division rings $D_1$ and $D_2$ having precisely the same set of subfields and such that $D_1$ is $k$-adequate (and so every subfield of $D_2$ is also $k$-adequate) but $D_2$ is not $k$-adequate.

Throughout this paper $K$ will denote an algebraic number field. We will use freely the classification theory of $K$-division algebras by means of Hasse invariants. The reader is referred to **[3]** for the relevant theory. If $\mathscr{P}$ is a prime of $K$ and $D$ is a $K$-division ring, we denote the Hasse invariant of $D$ at $\mathscr{P}$ by $\mathrm{inv}_{\mathscr{P}}\, D$. The order of $\mathrm{inv}_{\mathscr{P}}\, D$ in $Q/Z$ is denoted by $\mathrm{l.i.}_{\mathscr{P}}\, D$. Here $Q$ denotes the field of rational numbers and $Z$ is the ring of ordinary integers. We denote the completion of $K$ at the prime $\mathscr{P}$ by $K_{\mathscr{P}}$. The dimension of $D$ over $K$ is denoted by $[D:K]$; we use the same notation for the dimension of field extensions.

We begin by establishing criteria for two $K$-division rings to have precisely the same set of subfields.

THEOREM 1. *Let $D_1$ and $D_2$ be $K$-division rings. Then $D_1$ and $D_2$ have precisely the same set of subfields if and only if* $\mathrm{l.i.}_{\mathscr{P}}\, D_1 = \mathrm{l.i.}_{\mathscr{P}}\, D_2$ *for all primes $\mathscr{P}$ of $K$.*

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PROOF. Suppose $D_1$ and $D_2$ have precisely the same set of subfields but for some prime $\mathcal{P}$ of $K$ we have l.i.$_{\mathcal{P}}$ $D_1=n>m=$l.i.$_{\mathcal{P}}$ $D_2$. Using the Grünwald-Wang theorem [**2**, Theorem 5, p. 105] we can construct a cyclic extension $L$ of $K$ of degree $[D_2:K]^{1/2}$ such that $L$ is a splitting field for $D_2$ and such that the local degree of $L$ over $K$ at $\mathcal{P}$ equals $m$. It follows that $L$ is a subfield of $D_2$ [**1**, Theorem 27, p. 61] and so $L$ is a subfield of $D_1$. Since $D_1$ and $D_2$ have the same maximal subfields, $L$ is a maximal subfield of $D_1$. But then $n$ divides $m$, the local degree of $L$ over $K$ at $\mathcal{P}$ by [**4**, (0.1), p. 413]. This establishes the result in one direction. Conversely, suppose l.i.$_{\mathcal{P}}$ $D_1=$l.i.$_{\mathcal{P}}$ $D_2$ for all primes $\mathcal{P}$ of $K$. We need only check that $D_1$ and $D_2$ have the same set of maximal subfields. Let $L$ be a maximal subfield of $D_1$. Then l.i.$_{\mathcal{P}}$ $D_1$ divides $[L_{\mathcal{Y}}:K_{\mathcal{P}}]$ for every prime $\mathcal{Y}$ of $L$ extending $\mathcal{P}$ (see (0.1) of [**4**]). Thus $L$ also splits $D_2$.

Since the index of $D_1$ equals the least common multiple of $\{$l.i.$_{\mathcal{P}}$ $D_1\}$, it follows that $[D_1:K]=[D_2:K]$ and so $L$ is a maximal subfield of $D_2$. This proves the theorem.

COROLLARY 2. *Let $D_1$ and $D_2$ be quaternion algebras over $K$. Then $D_1\cong D_2$ if and only if $D_1$ and $D_2$ have precisely the same set of subfields.*

PROOF. Since $D_i$ is quaternion, l.i.$_{\mathcal{P}}$ $D_i=1$ or 2. Thus l.i.$_{\mathcal{P}}$ $D_1=$l.i.$_{\mathcal{P}}$ $D_2$ implies inv$_{\mathcal{P}}$ $D_1=$inv$_{\mathcal{P}}$ $D_2$ and the result now follows from [**3**, Satz 8, p. 119].

The following result is implicit in [**4**].

PROPOSITION 3. *Let $K$ be a normal extension of $k$ and let $D$ be a $K$-division ring. Suppose there is a prime $\mathcal{P}$ of $k$ such that* inv$_{\mathcal{Y}_1}$ $D\neq$inv$_{\mathcal{Y}_2}$ $D$ *for two primes $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of $K$ extending $\mathcal{P}$. Then $D$ is not $k$-adequate.*

PROOF. Suppose $D\subset D_0$, $D_0$ a $k$-division ring. As in the proof of [**4**, Theorem 1, p. 415], we have Cent$_{D_0}$ $(K)\cong D\otimes_K D'$, where Cent$_{D_0}$ $(K)$ is the centralizer of $K$ in $D_0$ and $D'$ is a $K$-division ring. By Lemma 1 of [**4**], Cent$_{D_0}$ $(K)\sim D_0\otimes_k K$. Since $K|k$ is normal, $[K_{\mathcal{Y}_1}:k_{\mathcal{P}}]=[K_{\mathcal{Y}_2}:k_{\mathcal{P}}]$. Thus inv$_{\mathcal{Y}_1}$ Cent$_{D_0}(K)=$inv$_{\mathcal{Y}_2}$ Cent$_{D_0}(K)$. This yields inv$_{\mathcal{Y}_1}$ $D'-$inv$_{\mathcal{Y}_2}$ $D'\equiv$ inv$_{\mathcal{Y}_2}$ $D-$inv$_{\mathcal{Y}_1}$ $D\not\equiv 0$ (mod 1). We conclude that for $i=1$ or 2, there is a prime $q$ dividing l.i.$_{\mathcal{Y}_i}$ $D'$ such that $q$ also divides l.i.$_{\mathcal{Y}_j}$ $D$ for some $j$. This contradicts (0.4) of [**4**].

We can now exhibit our example.

EXAMPLE. Let $K$ be a cyclic extension of $Q$ of degree 5 and let $p$ be a prime of $Q$ splitting completely in $K$. Let $r$, $s$, and $t$ be primes of $Q$ that remain prime in $K$. Such primes exist (see [**2**, Chapter 5, Theorem 3]). Let $D_1$ be the $K$-division ring having inv$_r$ $D_1=\frac{1}{5}$, inv$_s$ $D_1=\frac{4}{5}$, inv$_{\mathcal{P}}$ $D_1=\frac{1}{5}$ for all primes $\mathcal{P}$ of $K$ extending $p$, and inv$_{\mathcal{P}}$ $D_1=0$ for all other primes $\mathcal{P}$

of $K$. Let $D$ be the $Q$-division ring with $\text{inv}_p\, D=\frac{1}{5}$, $\text{inv}_r\, D=\frac{1}{25}$, $\text{inv}_s\, D=\frac{24}{25}$, $\text{inv}_t\, D=\frac{4}{5}$, and $\text{inv}_q\cdot D=0$ for all other primes $q$ of $Q$. These division algebras exist by [**3**, Satz 9, p. 119]. Let $L$ be a cyclic extension of $K$ of degree 5 having local degree 5 at all primes of $K$ extending $p$, $r$, and $s$. The existence of $L$ is guaranteed by the Grünwald-Wang theorem. Since $[D:Q]^{1/2}=25=[L:Q]$, $L$ is a maximal subfield of $D$ by (0.1) of [**4**]. Thus $K\subset D$. Let $\text{Cent}_D\,(K)$ be the centralizer of $K$ in $D$. This is a $K$-division ring by the double centralizer theorem and has the same invariants as $D_1$ by [**4**, Lemma 1, p. 413]. Thus $\text{Cent}_D\,(K)\cong D_1$ so $D_1$ is $Q$-adequate. Let $D_2$ be the $K$-division ring having $\text{inv}_r\, D_2=\frac{1}{5}$, $\text{inv}_s\, D_2=\frac{4}{5}$, $\text{inv}_{p_1}\, D_2=\frac{1}{5}$, $\text{inv}_{p_2}\, D_2=\frac{3}{5}$, $\text{inv}_{p_3}\, D_2=\text{inv}_{p_4}\, D_2=\text{inv}_{p_5}\, D_2=\frac{2}{5}$, where $p_1, p_2, p_3, p_4$, and $p_5$ are the primes of $K$ extending $p$, and $\text{inv}_{\mathscr{P}}\, D_2=0$ for all other primes $\mathscr{P}$ of $K$. By Theorem 1, $D_1$ and $D_2$ have precisely the same set of subfields. Proposition 3 shows that $D_2$ is not $Q$-adequate.

We note that the above argument can be generalized so as to yield two $K$-division rings $D_1$ and $D_2$ having the same subfields with only one of $D_1$ or $D_2$ being $k$-adequate whenever $K$ is a cyclic extension of $k$ of prime power degree $p^r$ with $p^r>2$.

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Department of Mathematics, Oregon State University, Corvallis, Oregon 97331