

ON A CHARACTERISATION OF MATRIX FUNCTIONS WHICH ARE DIFFERENCES OF TWO MONOTONE MATRIX FUNCTIONS¹

HARKRISHAN VASUDEVA

ABSTRACT. The class of matrix functions of 'bounded variation' was introduced by O. Dobsch in a paper published in 1937 [2]. The consideration of this class of functions immediately gives rise to the consideration of those matrix functions of order n on an interval $[a, b]$ that are representable as the difference of two monotone matrix functions on that interval. Such a difference will have high regularity properties when n is large and is therefore much more than simply a function of bounded variation. The characterization of this class was sought in the paper of Dobsch [2]. The purpose of this paper is to give a complete description of a related class: the functions defined on $(-1, 1)$ which have restrictions to any closed subinterval which are such differences.

1. Introduction. This paper concerns the study of the class of matrix functions of 'bounded variation' corresponding to the monotone functions introduced by Charles Loewner in a paper published in 1934 [3]. O. Dobsch tried to find a characterisation of functions given on a closed interval $[a, b]$ which were the differences of monotone matrix functions on that interval. He did not succeed in finding this characterisation, and neither have we. However, we give a complete description of the functions on an open interval which appear on any closed subinterval as the differences of two monotone matrix functions on that subinterval.

2. If A is an $n \times n$ real symmetric or Hermitian matrix, $f(A)$ is the matrix resulting from A by leaving the eigenvectors fixed while the corresponding eigenvalues λ are replaced by $f(\lambda)$. Thus, if $A = T'DT$ where T is a unitary matrix, T' its conjugate transpose, and D a diagonal matrix, then $F(A) = T'f(D)T$. The function $f(A)$ on all $n \times n$ Hermitian matrices with eigenvalues in the domain of $f(x)$ is called a matrix function of order n generated by f .

An operator function f associated with I is monotone provided

$$H \geq 0 \Rightarrow f(A + H) \geq f(A).$$

Received by the editors June 8, 1970.

AMS 1970 subject classifications. Primary 26A45.

¹ This paper contains the results of the doctoral thesis of the author. This work was supported in part by National Science Foundation Grants GP 5436-1328.

© American Mathematical Society 1972

When considering $n \times n$ Hermitian matrices, a monotone operator function is called a monotone matrix function of order n . A convenient summary of the results about monotone matrix functions may be found in [1]. Here we shall state a known result due to Dobsch [2].

THEOREM 1. *Let $f(x)$ be a real-valued function defined on an open interval $(-1, 1)$.² $f(x)$ is a monotone matrix function of order n in $(-1, 1)$ iff f is of class C^{2n-3} , its $(2n-3)$ rd derivative is convex and the matrix*

$$M'_n(x; f) = \begin{bmatrix} f'(x) & \frac{f''(x)}{2!} & \cdots & \frac{f^{(n)}(x)}{n!} \\ \frac{f''(x)}{2!} & \frac{f'''(x)}{3!} & & \frac{f^{(n+1)}(x)}{(n+1)!} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{f^{(n)}(x)}{n!} & \frac{f^{(n+1)}(x)}{(n+1)!} & & \frac{f^{(2n-1)}(x)}{(2n-1)!} \end{bmatrix}$$

(which makes sense almost everywhere) is nonnegative definite.

3. DEFINITION. Let $f(x)$ be a real-valued function defined on a closed interval $[a, b]$; $f(x)$ is called a matrix function of bounded variation if there exists a constant K such that

$$\sum_{v=1}^p \|f(A_v) - f(A_{v-1})\| \leq K$$

for all partitions $aI \leq A_0 \leq A_1 \leq \cdots \leq A_p \leq bI$.

Since the norms on the finite-dimensional space of matrices of order n are equivalent, it is not necessary to specify exactly what norm occurs in the definition. For the special case $n=1$, it is clear that the matrix functions of bounded variation coincide with the usual functions of bounded variation over $[a, b]$. An important theorem due to Dobsch [2] asserts that this is also the case for larger values of n .

Below we give a complete description of a class: the functions defined on $(-1, 1)$ whose restrictions to any closed subinterval generate matrix functions of order n which are differences of two monotone matrix functions.

² Throughout this paper the restriction to the interval $(-1, 1)$ is unessential. All results can easily be transformed to the case of an arbitrary open interval.

LEMMA 2. Let C be a compact subset of the $n \times n$ Hermitian matrices such that every element of C has its spectrum in $(-1, 1)$. Let $h(x) = \sqrt{x+2}$; then there exists a $K > 0$ so that the matrix

$$A + KM'_n(x; h)$$

is positive definite for every A in C and every x in $(-1, 1)$.

PROOF. Let $\|A\|$ be a norm on the space of matrices which is at least as large as the usual operator norm so that the spectrum of A is contained in the interval $(-\|A\|, \|A\|)$. This norm is clearly a continuous function on the compact C and bounded there by some positive μ . It follows that $-\mu I \leq A \leq \mu I$ for every A in C .

Consider next the family of matrices $M'_n(x; h)$ as x varies over the closed interval $[-1, 1]$. This is evidently a compact set of matrices depending continuously on x . Every matrix in the set is positive and so there exists a positive α so that $M'_n(x; h) \geq \alpha I$ for every x in $(-1, 1)$.

Note that if $\phi(x) = Kh(x)$ then $A + M'_n(x; \phi)$ is positive definite for all A in C and all x in $(-1, 1)$.

THEOREM 3. A real-valued function $f(x)$ defined on $(-1, 1)$ is the difference of two monotone matrix functions of order $n > 1$ on every closed subinterval $[a, b]$ if and only if f is C^{2n-3} , the derivative $f^{(2n-3)}(x)$ is absolutely continuous on $[a, b]$ and its derivative, $f^{(2n-2)}(x)$ is of bounded variation there.

PROOF. Let $[a, b]$ be an arbitrary but fixed closed subinterval of $(-1, 1)$. Then there exists a closed subinterval $[a', b']$ of $(-1, 1)$ such that $[a, b]$ is contained in the interior of $[a', b']$. If $f(x)$ is the difference of two monotone matrix functions of order $n > 1$ on $[a', b']$, then in view of Theorem 1, it follows that f is C^{2n-3} , the derivative $f^{(2n-3)}(x)$ is absolutely continuous and its derivative, $f^{(2n-2)}(x)$ is of bounded variation on at least $[a, b]$.

Suppose $f(x)$ is C^{2n-3} , the derivative $f^{(2n-3)}$ is absolutely continuous on $[a, b]$ and $f^{(2n-2)}(x)$ is of bounded variation there. We shall show that f can be expressed as the difference of two monotone functions of order n . Since $f^{(2n-2)}$ is of bounded variation, it follows that $f = G_1 - G_2$ where $G_i^{(2n-2)}$ ($i=1, 2$) are monotone increasing. The polynomial $p(x)$ which appears in the process of obtaining f from $f^{(2n-2)}$ may be absorbed into G_1 . Hence it is enough to prove the theorem when $f^{(2n-2)}$ is monotone increasing. To show that f is the difference of two monotone matrix functions, we must find a convenient $h \in P(-1, 1)$ ³ so that the matrix

³ $P(-1, 1)$ denotes the functions in the Pick class which are real and regular on the interval $(-1, 1)$ and which therefore admit analytic continuation into the lower half-plane which is given by reflection.

$M'_n(x; f+h)$ is a positive matrix for almost all $x \in [a, b]$. Setting $f(x)+h(x)=g(x)$, we obtain $f(x)=g(x)-h(x)$ where $g(x)$ and $h(x)$ are such that $M'_n(x; g)$ and $M'_n(x; h)$ are positive matrices for almost all $x \in [a, b]$. By a theorem of Dobsch [1, Theorem 2.4] and remarks preceding Lemma 2, it follows that $g(x)$ and $h(x)$ are monotone matrix functions of order n on $[a, b]$. We write $M'_n(x; f)=A(x)+B(x)$ where

$$B(x) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \frac{f^{(2n-1)}(x)}{(2n-1)!} \end{bmatrix}$$

and $A(x)$ is the remainder of $M'_n(x; f)$.

Since $A(x)$ does not contain $f^{(2n-1)}(x)$, it is a uniformly bounded matrix function of x in $[a, b]$. It follows from Lemma 2 that

$$A(x) + \lambda M'_n(x; h) > 0, \quad x \in [a, b],$$

if λ is sufficiently large. On the other hand $B(x) \geq 0$ almost everywhere, because $f^{(2n-1)}(x) > 0$ almost everywhere. It follows that

$$M'_n(x; f) + \lambda M'_n(x; h) > 0$$

almost everywhere, as required.

The author is indebted to the referee for simplifications of his original proof.

REFERENCES

1. J. Benda and S. Sherman, *Monotone and convex operator functions*, Trans. Amer. Math. Soc. **79** (1955), 58-71. MR **18**, 588.
2. O. Dobsch, *Matrixfunktionen beschränkter Schwankung*, Math. Z. **43** (1937), 353-388.
3. K. Loewner, *Über monotone Matrixfunktionen*, Math. Z. **38** (1934), 177-216.

CENTER FOR ADVANCED STUDY IN MATHEMATICS, PUNJAB UNIVERSITY, CHANDIGARH
14, INDIA