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THE CONVERSE TO A THEOREM OF SHARP ON GORENSTEIN MODULES

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ABSTRACT. Let A be a commutative local Noetherian ring with identity of Krull dimension n, m its maximal ideal. Sharp has proved that if A is Cohen-Macauley and a homomorphic image of a Gorenstein local ring, then A has a Gorenstein module M with $\dim_{A/m}\operatorname{Ext}^n(A/m, M)=1$. The aim of this note is to prove the converse to this theorem.

Throughout this note A will denote a commutative local Noetherian ring with identity; m will denote its maximal ideal. The concept of a Gorenstein module was introduced by Sharp in [**8**].

(1) DEFINITION. A nonzero finitely generated A-module M is called Gorenstein if the Cousin complex [**7**] provides a minimal injective resolution of M.

Sharp obtained various characterizations and properties of Gorenstein modules in [**8**]. In particular, he showed that for there to exist a Gorenstein A-module it is necessary that A be Cohen-Macaulay [**8**, (3.9)], and he showed that a Gorenstein A-module has zero annihilator [**8**, (4.12)]. It follows that (in the notation of [**8**]) if M is a Gorenstein module, then $\mu^i(m, M)=0$ if and only if $i\neq K\text{-dim } A$ [**8**, (3.11)]. We define the rank of the Gorenstein A-module M to be $\mu^n(m, M)$, where $n=K\text{-dim } A$. In [**9**] Sharp proved the following

(2) THEOREM. *If A is Cohen-Macaulay and a quotient of a Gorenstein local ring, then A has a Gorenstein module of rank* 1 [**9**, (3.1)].

The aim of this note is to prove the following converse:

(3) THEOREM. *If A has a Gorenstein module M of rank* 1, *then A is a quotient of a Gorenstein local ring.*

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The notation will be the same as that in [**8**], with the following exception: if M is a nonzero finitely generated A-module, the notation $\text{depth}_A M$ will be used instead of $\text{codh}_A M$.

(4) The principle of idealization, introduced by Nagata (see [**6**, p. 2]) will be our tool. From the ring A and an A-module N, we obtain a structure of a commutative ring with identity on the Cartesian product set $A\times N$. Addition is "componentwise", and multiplication is given by $(a_1, n_1)\cdot(a_2, n_2)=(a_1a_2, a_1n_2+n_1a_2)$. A is then a quotient ring of $A\times N$, for $A\times N/0\times N\simeq A$. Since $0\times N$ is nilpotent, all prime ideals in $A\times N$ are of the form $\mathfrak{p}\times N$ for some prime ideal $\mathfrak{p}$ of A. Hence $A\times N$ is also local, and $K\text{-dim } A\times N=K\text{-dim } A$. Using Cohen's theorem [**6**, (3.4)], we easily see that $A\times N$ is Noetherian if and only if N is finitely generated.

The following proposition can be deduced from [**5**, (10)]; however, for the sake of completeness we include a direct proof.

(5) Proposition. *Suppose the A-module N is nonzero. Then $A\times N$ is self-injective $\Leftrightarrow A$ is complete and $N\simeq E(A/m)$ (i.e. the injective envelope of the residue field of A).*

Proof. If A is complete and $N=E(A/m)$, then by [**2**, p. 30], $T=\text{Hom}_A(A\times N, N)$ is an injective $A\times N$-module, where $A\times N$ is regarded as an A-module by means of the natural ring homomorphism $A\to A\times N$. Now, by [**4**, (3.7)], the natural A-homomorphism $A\to\text{Hom}_A(N, N)$ is an isomorphism. Consequently, there result A-module isomorphisms $T\xrightarrow{\sim} N\oplus\text{Hom}_A(N, N)\xrightarrow{\sim} N\oplus A$, and a straightforward computation shows that the resulting isomorphism $T\to A\times N$ is actually an $A\times N$-isomorphism.

Conversely, if $A\times N$ is self-injective, then again by [**2**, p. 30], since $A\times N/0\times N\simeq A$, $\text{Hom}_{A\times N}(A\times N/0\times N, A\times N)\simeq\text{Ann}_A N\times N$ is an injective A-module. Since A is local and $N\neq 0$, $\text{Ann}_A N=0$. Furthermore the natural homomorphism $A\to\text{Hom}_A(N, N)$ is surjective. For let $f: N\to N$ be an A-homomorphism. The mapping $g: 0\times N\to A\times N$ given by $(0, n)\to(0, f(n))$ is an $A\times N$-homomorphism; hence, since $A\times N$ is self-injective, g can be extended to an $A\times N$-homomorphism $g': A\times N\to A\times N$. It follows that f is just multiplication by some element $a\in A$. Hence, since $\text{Ann}_A N=(0)$, N is an injective A-module for which the natural homomorphism $A\to\text{Hom}_A(N, N)$ is an isomorphism. Using [**4**, (3.7)] and the now established fact that the endomorphism ring of N is local, we conclude that $N\simeq E(A/m)$ and A is complete.

(6) Corollary. *Suppose A is an Artin local ring, $N\neq(0)$ an A-module. $A\times N$ is self-injective $\Leftrightarrow N\simeq E(A/m)$.*

We remark that a direct proof of the fact that if A is an Artin local ring, then $A\times E(A/m)$ is self-injective appears in [**3**, p. 14].

(7) THEOREM. *Suppose A is a Cohen-Macaulay ring, having Krull dimension n, and M a nonzero finitely generated A-module. Then $A \times M$ is Gorenstein $\Leftrightarrow$ M is a Gorenstein module of rank* 1.

PROOF. Assume first that M is a Gorenstein module of rank 1. Then by [**8**, (3.11)], $\operatorname{depth}_A M = \operatorname{depth} A = n$; hence we can find $(a_1, \cdots, a_n)$ an A-sequence and M-sequence (see [**8**, (1.7)]). Then an easy computation shows that $(a_1, 0), \cdots, (a_n, 0)$ is an $A \times M$-sequence, and

$$A \times M/((a_1, 0), \cdots, (a_n, 0)) \cong A/(a_1, \cdots, a_n) \times M/(a_1, \ldots, a_n)M = A' \times M'.$$

Since M is Gorenstein of rank 1, and $\mu_A^{n+i}(m, M) = \mu_{A'}^{i}(m', M')$ for all $i \geqq 0$ (see [**1**, (2.6)]), we find that M' is a Gorenstein A'-module of rank 1. Hence $M' \cong E(A'/m')$, since K-dim $A' = 0$ (see [**8**, (3.11)]). Now $A' \times M'$ is self-injective by (6), hence, again using [**1**, (2.6)], $A \times M$ is a Gorenstein ring.

Now assume conversely that $A \times M$ is Gorenstein. Let $k = \operatorname{depth}_A M \leqq n$. Let $(a_1, \cdots, a_k)$ be an A-sequence and M-sequence, and as before consider $A \times M/((a_1, 0), \cdots, (a_k, 0)) = A' \times M'$. If depth $(A' \times M') = n - k > 0$, choose an element (a', m') which is $A' \times M'$-regular. Then it is easily seen that a' must be M'-regular, a contradiction to the fact that $\operatorname{depth}_A M = k$. Hence depth $(A' \times M') = 0$, so that $n = k$; and $A' \times M'$ is self-injective. Then (6) implies that $M' = E(A'/m')$; hence M is a Gorenstein module of rank 1.

(8) COROLLARY.[2] *If A has a Gorenstein module M of rank* 1, *then A is a quotient of a Gorenstein local ring.*

Sharp has informed me that he has obtained the following extension of (2) for a commutative Noetherian ring B: If B is Cohen-Macaulay and is a quotient of a Gorenstein ring, then B has a Gorenstein module M for which $\mu^{ht\mathfrak{p}}(\mathfrak{p}, \Omega) = 1$ for all $\mathfrak{p} \in \operatorname{Spec} B$. The converse of this result can be obtained from (7) by straightforward use of localization. Combining the results of these investigations, we obtain the following

(9) COROLLARY. *Suppose B is a commutative Noetherian ring. Then there exists a Gorenstein B-module M having the property that $\mu^{ht\mathfrak{p}}(\mathfrak{p}, M) = 1$ for all $\mathfrak{p} \in \operatorname{Spec} B$ if and only if B is a Cohen-Macaulay ring which can be expressed as a homomorphic image of a Gorenstein (commutative Noetherian) ring.*

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[2] This result has been obtained independently by H. B. Foxby of Copenhagen.

References

1. H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28. MR **27** #3669.

2. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N.J., 1956. MR **17**, 1040.

3. T. H. Gulliksen, *Massey operations and the Poincaré series of certain local rings*, Preprint Series, No. 7, University of Oslo, Oslo, Norway, August 1970.

4. E. Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. **8** (1958), 511–528. MR **20** #5800.

5. B. J. Müller, *On Morita duality*, Canad. J. Math. **21** (1969), 1338–1347.

6. M. Nagata, *Local rings*, Interscience Tracts in Pure and Appl. Math., no. 13, Interscience, New York, 1962. MR **27** #5790.

7. R. Y. Sharp, *The Cousin complex for a module over a commutative Noetherian ring*, Math. Z. **112** (1969), 340–356.

8. ———, *Gorenstein modules*, Math. Z. **115** (1970), 117–139.

9. ———, *On Gorenstein modules over a complete Cohen-Macaulay ring*, Oxford Quart. J. Math. Oxford Ser. (2) **22** (1971).

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