

TWO NEW PROOFS OF LERCH'S FUNCTIONAL EQUATION

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ABSTRACT.

One bright Sunday morning I went to church,
 And there I met a man named Lerch.
 We both did sing in jubilation,
 For he did show me a new equation.

Two simple derivations of the functional equation of

$$\sum_{n=0}^{\infty} \exp[2\pi i n x] (n+a)^{-s}$$

are given. The original proof is due to Lerch.

If x is real and $0 < a \leq 1$, define

$$\varphi(x, a, s) = \sum_{n=0}^{\infty} \exp[2\pi i n x] (n+a)^{-s},$$

where $\sigma = \operatorname{Re} s > 1$ if x is an integer, and $\sigma > 0$ otherwise. Note that $\varphi(0, a, s) = \zeta(s, a)$, the Hurwitz zeta-function. Furthermore, if $a=1$, $\varphi(0, 1, s) = \zeta(s)$, the Riemann zeta-function.

In 1887, Lerch [1] derived the following functional equation for $\varphi(x, a, s)$.

THEOREM. *Let $0 < x < 1$. Then $\varphi(x, a, s)$ has an analytic continuation to the entire complex plane and is an entire function of s . Furthermore, for all s ,*

$$(1) \quad \begin{aligned} \varphi(x, a, 1-s) &= (2\pi)^{-s} \Gamma(s) \\ &\cdot \{ \exp[\frac{1}{2}\pi i s - 2\pi i a x] \varphi(-a, x, s) \\ &\quad + \exp[-\frac{1}{2}\pi i s + 2\pi i a(1-x)] \varphi(a, 1-x, s) \}. \end{aligned}$$

Lerch's proof [1] of (1) depends upon the evaluation of a certain loop integral. Our objective here is to give two simple, new proofs of (1). The first proof uses contour integration; the second employs the Euler-Maclaurin summation formula. By slight variations of our methods, one can derive the corresponding result, namely Hurwitz's formula, for $\varphi(0, a, s) = \zeta(s, a)$.

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FIRST PROOF. Assume that $s > 1$ is real. With the aid of Euler's integral representation for $\Gamma(s)$, it is easy to show that [1, pp. 19-20]

$$(2) \quad \Gamma(s)\varphi(x, a, s) = \int_0^\infty \frac{\exp[(1-a)u - 2\pi ix]}{\exp[u - 2\pi ix] - 1} u^{s-1} du.$$

If we put $x = b + \frac{1}{2}$, then $|b| < \frac{1}{2}$. Define

$$F(z) = \frac{\pi \exp[2\pi ibz]}{(z+a)^s \sin(\pi z)},$$

where the principal branch of $(z+a)^s$ is chosen. Choose c so that $-a < c < 0$. If m is a positive integer, let C_m denote the positively oriented contour consisting of the right half of the circle with center $(c, 0)$ and radius $m + \frac{1}{2} - c$ together with the vertical diameter through $(c, 0)$. By the residue theorem,

$$(3) \quad \frac{1}{2\pi i} \int_{C_m} F(z) dz = \sum_{n=0}^m \exp[2\pi inx](n+a)^{-s}.$$

Let Γ_m denote the circular part of C_m . Since $|b| < \frac{1}{2}$, there is a constant M , independent of m , such that for z on Γ_m , $m \geq 1$,

$$\left| \frac{\exp[2\pi ibz]}{\sin(\pi z)} \right| \leq M.$$

Hence,

$$\left| \int_{\Gamma_m} F(z) dz \right| \leq \frac{\pi^2(m + \frac{3}{2})M}{(m + \frac{1}{2})^s},$$

which tends to 0 as m tends to ∞ since $s > 1$. Thus, upon letting m tend to ∞ in (3), we find that

$$(4) \quad \begin{aligned} \varphi(x, a, s) &= -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(z) dz \\ &= \int_c^{c+i\infty} \frac{\exp[2\pi ibz - \pi iz]}{(z+a)^s(\exp[-2\pi iz] - 1)} dz \\ &\quad + \int_c^{c-i\infty} \frac{\exp[2\pi ibz + \pi iz]}{(z+a)^s(\exp[2\pi iz] - 1)} dz. \end{aligned}$$

We observe that the integrals in (4) converge uniformly in any compact set of the s -plane since $|b| < \frac{1}{2}$. Hence, (4) shows that $\varphi(x, a, s)$ can be analytically continued to an entire function of s .

Now suppose that $-1 < s < 0$. We wish to let c approach $-a$ in (4). In a neighborhood of $z = -a$, we have

$$\left| \frac{\exp[2\pi ibz \pm \pi iz]}{(z+a)^s(\exp[\pm 2\pi iz] - 1)} \right| \leq A |z+a|^{-s-1} \leq A |y|^{-s-1},$$

where A is some positive number. Since $-1 < s < 0$,

$$\int_0^{\mp 1} |y|^{-s-1} dy < \infty.$$

Hence, the two integrals on the right side of (4) converge uniformly on $-a \leq c \leq -a + \varepsilon$, for any number $\varepsilon > 0$. We may then let c tend to $-a$ in (4) to obtain

$$\begin{aligned} \varphi(x, a, s) &= i \exp[-\frac{1}{2}\pi is - 2\pi iab + \pi ia] \\ &\cdot \int_0^{\infty} \frac{\exp[-2\pi by + \pi y]}{y^s(\exp[2\pi y + 2\pi ia] - 1)} dy \\ &- i \exp[\frac{1}{2}\pi is - 2\pi iab - \pi ia] \\ &\cdot \int_0^{\infty} \frac{\exp[2\pi by + \pi y]}{y^s(\exp[2\pi y - 2\pi ia] - 1)} dy. \end{aligned}$$

If we make the substitutions $u = 2\pi y$ and $b = x - \frac{1}{2}$ and replace s by $1-s$, the above becomes, for $s > 1$,

$$\begin{aligned} \varphi(x, a, 1-s) &= \exp[\frac{1}{2}\pi is - 2\pi ia(x-1)](2\pi)^{-s} \\ &\cdot \int_0^{\infty} \frac{\exp[-u(x-1)]u^{s-1}}{\exp[u + 2\pi ia] - 1} du \\ &+ \exp[-\frac{1}{2}\pi is - 2\pi iax](2\pi)^{-s} \\ &\cdot \int_0^{\infty} \frac{\exp[ux]u^{s-1}}{\exp[u - 2\pi ia] - 1} du. \end{aligned}$$

If we now use (2), (1) immediately follows for $s > 1$. By analytic continuation, (1) is valid for all s .

SECOND PROOF. Let f have a continuous first derivative on $[c, m]$, where m is a positive integer. Then we have the Euler-Maclaurin summation formula,

$$(5) \quad \sum_{c < n \leq m} f(n) = \int_c^m f(u) du + \frac{1}{2}f(m) + (c - [c] - \frac{1}{2})f(c) \\ + \int_c^m (u - [u] - \frac{1}{2})f'(u) du.$$

Put $c = 1-a$ and $f(u) = (u+a)^{-s} \exp(2\pi iux)$, where $\sigma > 0$. Then, upon letting m tend to ∞ in (5), we obtain

$$(6) \quad \begin{aligned} \varphi(x, a, s) - a^{-s} &= \int_{1-a}^{\infty} (u+a)^{-s} \exp[2\pi iux] du \\ &+ (\frac{1}{2} - a) \exp[2\pi ix(1-a)] \\ &- s \int_{1-a}^{\infty} \frac{u - [u] - \frac{1}{2}}{(u+a)^{s+1}} \exp[2\pi iux] du \\ &+ 2\pi ix \int_{1-a}^{\infty} \frac{u - [u] - \frac{1}{2}}{(u+a)^s} \exp[2\pi iux] du. \end{aligned}$$

The three integrals on the right side of (6) all converge for $\sigma > 0$ by Dirichlet's test.

First, assume that $0 < \sigma < 1$. Since

$$(7) \quad [u] - u + \frac{1}{2} = \sum_{n=1}^{\infty} \frac{\sin(2\pi nu)}{\pi n}$$

if u is not an integer, we have formally

$$\begin{aligned} & \int_{-a}^{\infty} \frac{u - [u] - \frac{1}{2}}{(u + a)^s} \exp[2\pi i u x] du \\ &= -\frac{1}{\pi} \int_0^{\infty} u^{-s} \exp[2\pi i(u - a)x] \sum_{n=1}^{\infty} \frac{1}{n} \sin(2\pi n\{u - a\}) du \\ (8) \quad &= \frac{\exp[-2\pi i ax]}{2\pi i} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \exp[2\pi i na] \int_0^{\infty} u^{-s} \exp[2\pi i u(x - n)] du \right. \\ & \quad \left. - \exp[-2\pi i na] \int_0^{\infty} u^{-s} \exp[2\pi i u(x + n)] du \right\}. \end{aligned}$$

We must justify the inversion in order of summation and integration. Since the Fourier series in (7) is boundedly convergent, the inversion is justified if we integrate over $(0, b)$, where b is any finite number [2, p. 41]. We need then only show that, for $0 < \sigma < 1$,

$$(9) \quad \lim_{b \rightarrow \infty} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \exp[2\pi i na] \int_b^{\infty} u^{-s} \exp[2\pi i u(x - n)] du - \exp[-2\pi i na] \int_b^{\infty} u^{-s} \exp[2\pi i u(x + n)] du \right\} = 0.$$

Upon an integration by parts,

$$\begin{aligned} & \int_b^{\infty} u^{-s} \exp[2\pi i u(x - n)] du = O(b^{-\sigma}/n) \\ & \quad + \frac{s}{2\pi i(x - n)} \int_b^{\infty} u^{-s-1} \exp[2\pi i u(x - n)] du = O(b^{-\sigma}/n), \end{aligned}$$

as b tends to ∞ . By the same argument we obtain the same O -estimate for the integrals involving $\exp\{2\pi i u(x+n)\}$. Hence, (9) now easily follows.

Now, if $0 < \sigma < 1$ and $d \neq 0$ is real, we have [2, pp. 107-103]

$$(10) \quad \int_0^{\infty} u^{-s} \exp[idu] du = \Gamma(1 - s) |d|^{s-1} \exp[\frac{1}{2}\pi i(1 - s)\text{sgn } d].$$

Thus, (8) becomes

$$\begin{aligned}
 \int_{-a}^{\infty} \frac{u - [u] - \frac{1}{2}}{(u + a)^s} \exp[2\pi i u x] du & \\
 (11) \qquad \qquad \qquad &= -(2\pi)^{s-2} \Gamma(1 - s) \exp[\frac{1}{2}\pi i s - 2\pi i a x] \\
 &\cdot \sum_{n=1}^{\infty} \left\{ \frac{\exp[2\pi i n a]}{n(n - x)^{1-s}} + \frac{\exp[-\pi i s - 2\pi i n a]}{n(n + x)^{1-s}} \right\}.
 \end{aligned}$$

Using (10) again, we have, for $0 < \sigma < 1$,

$$\begin{aligned}
 \int_{-a}^{\infty} (u + a)^{-s} \exp[2\pi i u x] du & \\
 (12) \qquad \qquad \qquad &= \exp[-2\pi i a x] \int_0^{\infty} u^{-s} \exp[2\pi i u x] du \\
 &= \Gamma(1 - s)(2\pi x)^{s-1} \exp[\frac{1}{2}\pi i(1 - s) - 2\pi i a x].
 \end{aligned}$$

Hence, substituting (11) and (12) into (6), we obtain, for $0 < \sigma < 1$,

$$\begin{aligned}
 \varphi(x, a, x) - a^{-s} & \\
 &= \Gamma(1 - s)(2\pi x)^{s-1} \exp[\frac{1}{2}\pi i(1 - s) - 2\pi i a x] \\
 &\quad - \int_{-a}^{1-a} (u + a)^{-s} \exp[2\pi i u x] \{1 + 2\pi i x(u - [u] - \frac{1}{2})\} du \\
 (13) \qquad \qquad \qquad &+ (\frac{1}{2} - a) \exp[2\pi i x(1 - a)] - s \int_{1-a}^{\infty} \frac{u - [u] - \frac{1}{2}}{(u + a)^{s+1}} \exp[2\pi i u x] du \\
 &\quad + x(2\pi)^{s-1} \Gamma(1 - s) \exp[\frac{1}{2}\pi i(s - 1) - 2\pi i a x] \\
 &\quad \cdot \sum_{n=1}^{\infty} \left\{ \frac{\exp[2\pi i n a]}{n(n - x)^{1-s}} + \frac{\exp[-\pi i s - 2\pi i n a]}{n(n + x)^{1-s}} \right\}.
 \end{aligned}$$

We next observe that the infinite series on the right side of (13) converges absolutely and uniformly on any compact subset of the strip $-1 < \sigma < 1$. By Dirichlet's test, the integrals on the right side of (13) converge uniformly on any compact subset of the strip $-1 < \sigma < 1$. Hence, by analytic continuation, (13) is valid for $-1 < \sigma < 1$. Assume now that $-1 < \sigma < 0$. Replacing s by $s+1$ in (11), we have

$$\begin{aligned}
 \int_{-a}^{\infty} \frac{u - [u] - \frac{1}{2}}{(u + a)^{s+1}} \exp[2\pi i u x] du & \\
 (14) \qquad \qquad \qquad &= (2\pi)^{s-1} \Gamma(-s) \exp[\frac{1}{2}\pi i(s - 1) - 2\pi i a x] \\
 &\quad \cdot \sum_{n=1}^{\infty} \left\{ \frac{\exp[2\pi i n a]}{n(n - x)^{-s}} - \frac{\exp[-i\pi s - 2\pi i n a]}{n(n + x)^{-s}} \right\}.
 \end{aligned}$$

Substitute (14) into (13), use the functional equation of $\Gamma(s)$, and observe that $x/\{n(n-x)\} + 1/n = 1/(n-x)$ and $x/\{n(n+x)\} - 1/n = -1/(n+x)$. For $-1 < \sigma < 0$ we arrive at

$$\begin{aligned}
 \varphi(x, a, s) &= a^{-s} \\
 &= \Gamma(1-s)(2\pi x)^{s-1} \exp[\tfrac{1}{2}\pi i(1-s) - 2\pi iax] \\
 &\quad - \int_{-a}^{1-a} (u+a)^{-s} \exp[2\pi iux] \\
 (15) \quad &\cdot \{1 + 2\pi ix(u - [u] - \tfrac{1}{2}) - s(u+a)^{-1}(u - [u] - \tfrac{1}{2})\} du \\
 &\quad + (\tfrac{1}{2} - a) \exp[2\pi ix(1-a)] \\
 &\quad + (2\pi)^{s-1} \Gamma(1-s) \exp[\tfrac{1}{2}\pi i(s-1) - 2\pi iax] \\
 &\quad \cdot \left\{ \sum_{n=1}^{\infty} \frac{\exp[2\pi ina]}{(n-x)^{1-s}} - \sum_{n=1}^{\infty} \frac{\exp[-\pi is - 2\pi ina]}{(n+x)^{1-s}} \right\}.
 \end{aligned}$$

Observe that the first expression on the right side of (15) corresponds to the term $n=0$ for the second series on the right side of (15). In the first series replace n by $n+1$. By an elementary calculation the second expression on the right side of (15) is seen to be

$$-a^{-s} - (\tfrac{1}{2} - a) \exp[2\pi ix(1-a)].$$

Upon these simplifications, (15) now becomes, for $-1 < \sigma < 0$,

$$\begin{aligned}
 \varphi(x, a, x) \\
 (16) \quad &= (2\pi)^{s-1} \Gamma(1-s) \exp[\tfrac{1}{2}\pi i(s-1) - 2\pi iax] \\
 &\quad \cdot \{ \exp[2\pi ia] \varphi(a, 1-x, 1-s) - \exp[-\pi is] \varphi(-a, x, 1-s) \}.
 \end{aligned}$$

By analytic continuation (16) is valid for all s . Now replace s by $1-s$ in (16) to obtain (1).

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