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CONTINUOUS DEPENDENCE ON A IN THE D_1AD_2 THEOREMS

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ABSTRACT. It has been shown by Sinkhorn and Knopp and others that if A is a nonnegative square matrix such that there exists a doubly stochastic matrix B with the same zero pattern as A, then there exists a unique doubly stochastic matrix of the form D_1AD_2 where D_1 and D_2 are diagonal matrices with positive main diagonals. Sinkhorn and Knopp have also shown that if A has at least one positive diagonal, then the sequence of matrices obtained by alternately normalizing the row and column sums of A will converge to a doubly stochastic limit. It is the intent of this paper to show that D_1AD_2 and/or the limit of this iteration, when either exists, is continuously dependent upon the matrix A.

Introduction. An $N\times N$ matrix $A=(a_{ij})$ is said to be nonnegative if every $a_{ij}\geqq 0$. For such a matrix A we write $A\geqq 0$.

An $N\times N$ matrix $A=(a_{ij})$ is said to be doubly stochastic if $A\geqq 0$ and if $\sum_{k=1}^N a_{ik}=\sum_{k=1}^N a_{kj}=1$ for all i and j. The set of $N\times N$ doubly stochastic matrices is denoted by Ω_N.

We say that the $N\times N$ nonnegative matrices A and B have the same pattern if $a_{ij}=0\Leftrightarrow b_{ij}=0$. We say that the $N\times N$ nonnegative matrix A has a subpattern of an $N\times N$ nonnegative matrix B if $a_{ij}=0\Rightarrow b_{ij}=0$. If A is an $N\times N$ nonnegative matrix such that there exists a $B\in\Omega_N$ with the same pattern as A, we say that A has doubly stochastic pattern. The set of all $N\times N$ nonnegative matrices with doubly stochastic pattern is denoted by $\mathscr{P}(\Omega_N)$. If A is an $N\times N$ nonnegative matrix and if there exists a $B\in\Omega_N$ such that A has a subpattern of B, we say that A has doubly stochastic subpattern. The set of all $N\times N$ nonnegative matrices with doubly stochastic subpattern is denoted by $\mathscr{S}(\Omega_N)$. Observe that $\mathscr{P}(\Omega_N)\subseteq\mathscr{S}(\Omega_N)$.

We denote by S_N the set of all permutations of $1,\cdots,N$. If A is an $N\times N$ matrix and $\sigma\in S_N$, the set of elements $a_{1\sigma(1)},\cdots,a_{N\sigma(N)}$ is called a diagonal of A. If every $a_{i\sigma(i)}>0$, we say that the diagonal is positive. In case σ is the identity permutation, we call the diagonal the main diagonal of A.

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If A is an $N \times N$ matrix, we define the permanent of A, per A, by

$$\text{per } A = \sum_{\sigma \in S_N} \prod_{i=1}^{N} a_{i\sigma(i)}.$$

In [2], Sinkhorn and Knopp show that if $A \in \mathscr{P}(\Omega_N)$, there exists a unique matrix of the form $D_1 A D_2 \in \Omega_N$ where D_1 and D_2 are diagonal matrices with positive main diagonals. They show that $D_1 A D_2$ is the limit of a sequence of matrices obtained by alternately scaling the rows and columns of A. They show in fact that this matrix sequence will converge to a limit in Ω_N if and only if $A \in \mathscr{S}(\Omega_N)$. The limit has the form $D_1 A D_2$, however, only if $A \in \mathscr{P}(\Omega_N)$.

It is the intent of this paper to show that $D_1 A D_2$ is a continuous function of A on $\mathscr{P}(\Omega_N)$ and that the limit of the iteration is a continuous function of A on $\mathscr{S}(\Omega_N)$. The following result of Sinkhorn and Knopp [3] is the main tool in the development.

THEOREM 1. *Distinct $N \times N$ doubly stochastic matrices A and B do not have proportional corresponding diagonal products, i.e. there is no $k > 0$ such that for each $\sigma \in S_N$, $\prod_{i=1}^{N} a_{i\sigma(i)} = k \prod_{i=1}^{N} b_{i\sigma(i)}$.*

The following celebrated theorem of G. Birkhoff may be employed to show that per $A > 0$ for any $A \in \Omega_N$. For a proof see [1, p. 98].

THEOREM 2. *The set of all $N \times N$ doubly stochastic matrices forms a convex polyhedron with the permutation matrices as vertices.*

Results and consequences.

THEOREM 3. *Suppose that $A \geqq 0$ is an $N \times N$ matrix such that* per $A > 0$. *Suppose that for each positive integer n and $k = 1, 2$, $A_k(n) \geqq 0$ is an $N \times N$ matrix such that for every permutation $\sigma \in S_N$, $\lim_{n \to \infty} \prod_{i=1}^{N} a_{ki\sigma(i)}(n) = \prod_{i=1}^{N} a_{i\sigma(i)}$. If for each positive integer n and $k = 1, 2$, $E_k(n)$ and $F_k(n)$ are $N \times N$ diagonal matrices with positive main diagonals such that*

$$\lim_{n \to \infty} E_k(n) A_k(n) F_k(n) = P_k \in \Omega_N,$$

then necessarily $P_1 = P_2$.

PROOF. For $k = 1, 2$, put $K_k = \text{per } P_k / \text{per } A$. Since each $P_k \in \Omega_N$, per $P_k > 0$ and so each $K_k > 0$. It is seen that for each such value of k, per $A_k(n) \to$ per A, and thus

$$\lim_{n \to \infty} \text{per } E_k(n) F_k(n) = \lim_{n \to \infty} \frac{\text{per } E_k(n) A_k(n) F_k(n)}{\text{per } A_k(n)} = \frac{\text{per } P_k}{\text{per } A} = K_k.$$

Let $\Delta_A \subseteq S_N$ denote those permutations σ such that $a_{1\sigma(1)}, \cdots, a_{N\sigma(N)}$ is a positive diagonal in A. Since per $A > 0$, $\Delta_A \neq \varnothing$.

Put $P_k=(p_{kij})$ for $k=1, 2$. Then for either value of k and any $\sigma\in\Delta_A$,

$$\prod_{i=1}^{N}\frac{p_{ki\sigma(i)}}{a_{i\sigma(i)}}=\lim_{n\to\infty}\frac{[\operatorname{per}E_k(n)F_k(n)]\prod_{i=1}^{N}a_{ki\sigma(i)}(n)}{\prod_{i=1}^{N}a_{i\sigma(i)}}=K_k,$$

where $A_k(n)=(a_{kij}(n))$.

From

$$\begin{aligned}\operatorname{per}P_k &= \sum_{\sigma\in S_N}\prod_{i=1}^{N}p_{ki\sigma(i)}=\sum_{\sigma\in\Delta_A}\prod_{i=1}^{N}p_{ki\sigma(i)}+\sum_{\sigma\in S_N-\Delta_A}\prod_{i=1}^{N}p_{ki\sigma(i)}\\ &= K_k\sum_{\sigma\in\Delta_A}\prod_{i=1}^{N}a_{i\sigma(i)}+\sum_{\sigma\in S_N-\Delta_A}\prod_{i=1}^{N}p_{ki\sigma(i)}\\ &= K_k\operatorname{per}A+\sum_{\sigma\in S_N-\Delta_A}\prod_{i=1}^{N}p_{ki\sigma(i)}=\operatorname{per}P_k+\sum_{\sigma\in S_N-\Delta_A}\prod_{i=1}^{N}p_{ki\sigma(i)},\end{aligned}$$

we see that $\prod_{i=1}^{N}p_{ki\sigma(i)}=0$ for each $\sigma\in S_N-\Delta_A$, $k=1, 2$. Since $\prod_{i=1}^{N}a_{i\sigma(i)}=0$ for each $\sigma\in S_N-\Delta_A$, it follows that $\prod_{i=1}^{N}p_{1i\sigma(i)}=K_1\prod_{i=1}^{N}a_{i\sigma(i)}=(K_1/K_2)\prod_{i=1}^{N}p_{2i\sigma(i)}$ for *all* $\sigma\in S_N$. Thus, by Theorem 1, $P_1=P_2$.

COROLLARY 1. *Let $A\geqq 0$ be an $N\times N$ matrix such that* per $A>0$. *Put $\bar{A}=(\bar{a}_{ij})$ where $\bar{a}_{ij}=a_{ij}$ if a_{ij} lies on at least one positive diagonal in A, and $\bar{a}_{ij}=0$ otherwise. Let D_1 and D_2 be diagonal matrices with positive main diagonals such that $D_1\bar{A}D_2\in\Omega_N$. Then the limit of the iteration of alternately normalizing the row and column sums of A is equal to $D_1\bar{A}D_2$.*

PROOF. Let the nth term of the iteration be denoted by $E_1(n)A_1(n)F_1(n)$ where $A_1(n)\equiv A$. Also put $E_2(n)=D_1$, $A_2(n)=\bar{A}$, and $F_2(n)=D_2$ for all n. It follows from the Sinkhorn-Knopp result [2] that $\lim_{n\to\infty}E_1(n)A_1(n)F_1(n)$ exists. Since $\prod_{i=1}^{N}a_{ki\sigma(i)}(n)=\prod_{i=1}^{N}a_{i\sigma(i)}=\prod_{i=1}^{N}\bar{a}_{i\sigma(i)}$ for any $\sigma\in S_N$, Theorem 3 shows that this limit is in fact $D_1\bar{A}D_2$.

COROLLARY 2. *Let $A\geqq 0$ be an $N\times N$ matrix such that* per $A>0$. *For any $\varepsilon>0$ suppose that $A(\varepsilon)\geqq 0$ is an $N\times N$ matrix such that*

$$\lim_{\varepsilon\downarrow 0}\prod_{i=1}^{N}a_{i\sigma(i)}(\varepsilon)=\prod_{i=1}^{N}a_{i\sigma(i)}$$

for every $\sigma\in S_N$. Let $\bar{A}$, D_1, and D_2 be as in Corollary 1, *and let $\bar{A}(\varepsilon)$, $D_1(\varepsilon)$, and $D_2(\varepsilon)$ be the corresponding matrices for $A(\varepsilon)$ whenever* per $A(\varepsilon)>0$. *Then*

$$\lim_{\varepsilon\downarrow 0}D_1(\varepsilon)\bar{A}(\varepsilon)D_2(\varepsilon)=D_1\bar{A}D_2.$$

PROOF. Observe that $D_1(\varepsilon)\bar{A}(\varepsilon)D_2(\varepsilon)$ exists for ε sufficiently small since for such values of ε, per $A(\varepsilon)>0$.

Since Ω_N is compact, the set $\{D_1(\varepsilon)\bar{A}(\varepsilon)D_2(\varepsilon)\}$ is bounded and therefore has at least one limit point as $\varepsilon\downarrow 0$. Of course all the limit points of this set are doubly stochastic. Moreover, since

$$\lim_{\varepsilon\downarrow 0}\prod_{i=1}^{N}\bar{a}_{i\sigma(i)}(\varepsilon) = \lim_{\varepsilon\downarrow 0}\prod_{i=1}^{N}a_{i\sigma(i)}(\varepsilon) = \prod_{i=1}^{N}a_{i\sigma(i)} = \prod_{i=1}^{N}\bar{a}_{i\sigma(i)}$$

for all $\sigma\in S_N$, any two convergent subsequences of $\{D_1(\varepsilon)\bar{A}(\varepsilon)D_2(\varepsilon)\}$ must have the same limit by Theorem 3. Thus the set $\{D_1(\varepsilon)\bar{A}(\varepsilon)D_2(\varepsilon)\}$ has exactly one limit point as $\varepsilon\downarrow 0$ and so $\lim_{\varepsilon\downarrow 0} D_1(\varepsilon)\bar{A}(\varepsilon)D_2(\varepsilon)=P$ exists. Clearly $P\in\Omega_N$.

Put $E_1(n)=D_1(1/n)$, $A_1(n)=\bar{A}(1/n)$, and $F_1(n)=D_2(1/n)$ and put $E_2(n)=D_1$, $A_2(n)=\bar{A}$, and $F_2(n)=D_2$ for $n=1, 2, \cdots$. By Theorem 3,

$$\lim_{n\to\infty} E_1(n)A_1(n)F_1(n) = P = D_1\bar{A}D_2 = \lim_{n\to\infty} E_2(n)A_2(n)F_2(n).$$

Whence

$$\lim_{\varepsilon\downarrow 0} D_1(\varepsilon)\bar{A}(\varepsilon)D_2(\varepsilon) = D_1\bar{A}D_2.$$

Corollaries 3 and 4 which follow are immediate consequences of Corollaries 1 and 2.

COROLLARY 3. *The limit of the iteration of alternately normalizing the row and column sums of an* $N\times N$ *matrix* A *is a continuous function of* A *on* $\mathscr{S}(\Omega_N)$.

COROLLARY 4. *For each* $A\in\mathscr{P}(\Omega_N)$ *there is a unique matrix* $D_1AD_2\in\Omega_N$ *where* D_1 *and* D_2 *are diagonal matrices with positive main diagonals. The map* $A\to D_1AD_2$ *is continuous on* $\mathscr{P}(\Omega_N)$.

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