PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 2, April 1972

DIRICHLET FINITE SOLUTIONS OF $\Delta u=Pu$

IVAN J. SINGER[1]

ABSTRACT. The purpose of this paper is to give a necessary and also a sufficient condition for a Dirichlet finite harmonic function on a Riemann surface to be represented as a difference of a Dirichlet finite solution of $\Delta u=Pu$ ($P\geqq 0$) and a Dirichlet finite potential of signed measure.

1. Let $P=P(z)\,dx\,dy$ ($z=x+iy$) be a nonnegative not identically zero α-Hölder continuous ($0<\alpha\leqq 1$) second order differential on a Riemann surface R and $\mathrm{PD}(R)$ be the Hilbert space of all Dirichlet finite solutions of

$$(1)\qquad \Delta u(z) = P(z)u(z), \qquad \Delta\cdot = 4\partial^2\cdot/\partial z\partial\bar{z},$$

on R with the scalar product given by mixed Dirichlet integral, i.e. $(u,v)=D_R(u,v)=\int_R du\wedge *dv$, *not* the energy integral.

The study of $\mathrm{PD}(R)$ was begun by Royden [**6**]. We will use the fact shown by Nakai [**2**] that $\mathrm{PD}(R)$ forms a vector lattice under the natural order in $\mathrm{PD}(R)$. We also use the Glasner-Katz maximum principle [**1**] that the modulus of every function in $\mathrm{PD}(R)$ takes its maximum on the Royden harmonic boundary. The recent result of Nakai [3] that $\mathrm{PBD}(R)$ is dense in $\mathrm{PD}(R)$ will not be made use of.

Let $\Delta(R)$ be the Royden harmonic boundary and $\mathrm{HD}(R)$ be the class of Dirichlet finite harmonic functions on R. (For the basic materials from the Royden compactification and the class $\mathrm{HD}(R)$ we refer to the monograph of Sario and Nakai [7].) One of the important problems in the theory of $\mathrm{PD}(R)$ which is not fully developed yet is to describe the distribution of $\mathrm{PD}(R)|\Delta(R)$ in $\mathrm{HD}(R)|\Delta(R)$. We will prove a theorem which contributes to this question.

2. If R is parabolic, then $\mathrm{PD}(R)=\{0\}$ (cf. Royden [**6**]), which case offers no interest. Therefore we assume throughout the paper that R is hyperbolic. Let $\tilde{M}(R)$ be the class of all Dirichlet finite Tonelli functions on R and $\tilde{M}_\Delta(R)$ the subclass of $\tilde{M}(R)$ consisting of functions f with $f|\Delta(R)=0$ (cf. [7]). We then have the orthogonal decomposition

$$\tilde{M}(R) = \mathrm{HD}(R) + \tilde{M}_\Delta(R),$$

Received by the editors May 26, 1971.

AMS 1970 *subject classifications*. Primary 31A05, 58G99.

[1] The work was sponsored by the U.S. Army Research Office—Durham, Grant DA-ARO-D-31-124-71-G20, UCLA.

and since $\mathrm{PD}(R)\subset\tilde{M}(R)$, we can define an operator $T:\mathrm{PD}(R)\to\mathrm{HD}(R)$ characterized by

$$u - Tu \in \tilde{M}_{\Delta}(R). \tag{2}$$

Using results cited in §1 we can show that T is a vector space isomorphism from $\mathrm{PD}(R)$ onto

$$X_D(R) \equiv T(\mathrm{PD}(R)) \tag{3}$$

such that $u>0$ is equivalent to $Tu>0$ and $\sup_R |u|=\sup_R |Tu|$. Therefore the study of $\mathrm{PD}(R)$ can be reduced to that of $X_D(R)$ and for this reason we call T the *reduction operator* for Dirichlet finite solutions. It can be seen that

$$u = Tu - \frac{1}{2\pi}\int_R G(\cdot,\zeta)P(\zeta)u(\zeta)\,d\xi\,d\eta \qquad (\zeta=\xi+i\eta)$$

(cf. [**3**]). We will discuss when $h\in\mathrm{HD}(R)$ belongs to $X_D(R)$.

3. Let Ω be a regular subregion of R. By the *P-unit* e_Ω on Ω we mean the solution e_Ω of (1) on Ω with the continuous boundary values 1. The net $\{e_\Omega\}$ for every regular subregion Ω is decreasing and hence convergent to a solution on R:

$$e_R = \lim_{\Omega\to R} e_\Omega \geqq 0$$

which we call the P-unit on R. The only bounded solution of (1) on R is zero if and only if $e_R\equiv 0$ (Ozawa [**5**], Royden [**6**]). We describe $X_D(R)$ in terms of $\{e_\Omega\}$ and e_R as follows:

THEOREM. *Suppose that $h\in\mathrm{HD}(R)$. If $h\in X_D(R)$, then*

$$D_R(e_R h) < \infty. \tag{4}$$

Conversely if

$$\limsup_{\Omega\to R} D_\Omega(e_\Omega h) < \infty, \tag{5}$$

then $h\in X_D(R)$.

The proof will be given in §§4 and 7. The condition (4) is necessary for $h\in X_D(R)$ but not sufficient. In fact, let $R=\{|z|<1\}$ and $P(z)=4(1+|z|^2)(1-|z|^2)^{-2}$. Then $e_R\equiv 0$ and $X_D(R)=\{0\}$ (Royden [**6**]), while (4) is trivially valid for every $h\in\mathrm{HD}(R)$. The condition (5) is sufficient for $h\in X_D(R)$ but not necessary. We exhibit an instructive example due to Nakai [**4**]. Let $R=\{|z|>1\}$ and $P(z)=1+|z|^{-1}$. Consider $\Omega_n=\{1+n^{-1}<|z|<n\}$ $(n=2,3,\cdots)$ which exhausts R as $n\to\infty$. Denote by e_n the P-unit on Ω_n. The P-unit e_R on R is given by

$$e_R(z) = \left(e\int_1^\infty e^{-2t}t^{-1}\,dt\right)^{-1}\cdot e^{|z|}\cdot\int_{|z|}^\infty e^{-2t}\cdot t^{-1}\,dt$$

and a straightforward calculation shows that $e_R \in \mathrm{PD}(R)$ and hence $1 \in X_D(R)$. We also see that

$$e_n(z) = \alpha_n e^{|z|} + \beta_n e^{|z|} \int_1^{|z|} e^{-2t} t^{-1}\, dt$$

where

$$\alpha_n = e^{-n} - (e^{-n} - e^{-(1+1/n)}) \left(\int_{1+1/n}^{n} e^{-2t} t^{-1}\, dt \right)^{-1} \int_1^n e^{-2t} t^{-1}\, dt,$$

and

$$\beta_n = (e^{-n} - e^{-(1+1/n)}) \left(\int_{1+1/n}^{n} e^{-2t} t^{-1}\, dt \right)^{-1}.$$

However an easy but cumbersome computation shows that

$$D_R(1 \cdot e_n) = \mathcal{O}(n) \qquad (n \to \infty)$$

and (5) is not valid for $h = 1 \in X_D(R)$. By Fatou's lemma, (5) implies (4). That the converse is not necessarily true is also seen from the above example.

4. **Necessity of (4).** Suppose $h \in X_D(R)$. Since $X_D(R)$ forms a vector lattice along with $\mathrm{PD}(R)$, we may assume $h > 0$ to prove (4). Let $u \in \mathrm{PD}(R)$ such that $h = Tu$ and let $\varphi = u - h$. We will prove (4) both for u and φ.

We write $\|\cdot\|_\Omega = (D_\Omega(\cdot))^{1/2}$. By Green's formula

$$\begin{aligned}
\|u(1-e_\Omega)\|_\Omega^2 &= -\int_\Omega u(1-e_\Omega) d\, {}^*d(u(1-e_\Omega)) \\
&= -\int_\Omega u^2(1-e_\Omega)^2 P + \int_\Omega u^2(1-e_\Omega) e_\Omega P \\
&\quad + 2\int_\Omega u(1-e_\Omega)\, du \wedge {}^*de_\Omega \\
&\leqq \int_\Omega u(1-e_\Omega) Pu + 2\int_\Omega (1-e_\Omega)\, du \wedge {}^*(u\, de_\Omega) \\
&= \int_\Omega u(1-e_\Omega) d\, {}^*du + 2\int_\Omega (1-e_\Omega)\, du \wedge {}^*(d(ue_\Omega) - e_\Omega\, du).
\end{aligned}$$

Observe that $\int_\Omega u(1-e_\Omega) d\, {}^*du = -\int_\Omega d(u(1-e_\Omega)) \wedge {}^*du$. By Schwarz's inequality

$$\|u(1-e_\Omega)\|_\Omega^2 \leqq \|u(1-e_\Omega)\|_\Omega \, \|u\|_\Omega + 2\, \|u\|_\Omega \, \|ue_\Omega\|_\Omega + 2\, \|u\|_\Omega^2.$$

In view of $\|ue_\Omega\|_\Omega \leqq \|u(1-e_\Omega)\|_\Omega + \|u\|_\Omega$, we deduce

$$\|u(1-e_\Omega)\|_\Omega^2 \leqq 3\, \|u(1-e_\Omega)\|_\Omega \cdot \|u\|_\Omega + 4\, \|u\|_\Omega^2.$$

This implies $\|u(1-e_\Omega)\|_\Omega \leqq 4\|u\|_\Omega$ or $\|ue_\Omega\|_\Omega \leqq 5\|u\|_\Omega$. Therefore, by Fatou's lemma,

$$D_R(e_R u) \leqq \liminf_{\Omega \to R} D_\Omega(e_\Omega u) \leqq 25 D_R(u) < \infty.$$

5. Let $h_\Omega \in C(\bar{\Omega})$ such that h_Ω is harmonic in Ω and $h_\Omega|\partial\Omega=u$. Set $\varphi_\Omega=u-h_\Omega$. Observe $\Delta\varphi_\Omega=Pu$ and $\varphi_\Omega\leqq 0$. Since $D_\Omega(u)=D_\Omega(h_\Omega)+D_\Omega(\varphi_\Omega)$ and $\lim_{\Omega\to R} h_\Omega=h$, we infer that $\varphi=\lim_{\Omega\to R}\varphi_\Omega$, $d\varphi=\lim_{\Omega\to R} d\varphi_\Omega$, and $D_\Omega(\varphi_\Omega)\leqq D_\Omega(u)$. By Green's formula,

$$\begin{aligned}\|e_\Omega\varphi_\Omega\|_\Omega^2 &= -\int_\Omega e_\Omega\varphi_\Omega d\,{}^*d(e_\Omega\varphi_\Omega)\\ &= -\int_\Omega e_\Omega^2\varphi_\Omega^2 P - \int_\Omega e_\Omega^2\varphi_\Omega uP - 2\int_\Omega e_\Omega\varphi_\Omega\, de_\Omega \wedge {}^*d\varphi_\Omega\\ &\leqq -\int_\Omega \varphi_\Omega d\,{}^*du - 2\int_\Omega e_\Omega\, d\varphi_\Omega \wedge {}^*(\varphi_\Omega\, de_\Omega)\\ &= \int_\Omega d\varphi_\Omega \wedge {}^*du - 2\int_\Omega e_\Omega\, d\varphi_\Omega \wedge {}^*(d(e_\Omega\varphi_\Omega) - e_\Omega\, d\varphi_\Omega).\end{aligned}$$

By Schwarz's inequality,

$$\begin{aligned}\|e_\Omega\varphi_\Omega\|_\Omega^2 &\leqq \|\varphi_\Omega\|_\Omega\,\|u\|_\Omega + 2\,\|\varphi_\Omega\|_\Omega\,\|e_\Omega\varphi_\Omega\|_\Omega + 2\,\|\varphi_\Omega\|_\Omega^2\\ &\leqq 2\,\|u\|_\Omega\,\|e_\Omega\varphi_\Omega\|_\Omega + 3\,\|u\|_\Omega^2\end{aligned}$$

and therefore $\|e_\Omega\varphi_\Omega\|_\Omega \leqq 3\,\|u\|_\Omega$. By Fatou's lemma we deduce

$$D_R(e_R\varphi) \leqq \liminf_{\Omega\to R} D_\Omega(e_\Omega\varphi_\Omega) \leqq 9D_R(u) < \infty.$$

6. **Sufficiency of (5).** Let $u_\Omega \in C(\bar{\Omega})$ such that $\Delta u_\Omega(z)=P(z)u_\Omega(z)$ on Ω and $u_\Omega|\partial\Omega=h$. By Green's formula,

$$\begin{aligned}\|u_\Omega - he_\Omega\|_\Omega^2 &= -\int_\Omega (u_\Omega - he_\Omega)d\,{}^*d(u_\Omega - he_\Omega)\\ &= -\int_\Omega (u_\Omega - he_\Omega)u_\Omega P + \int_\Omega (u_\Omega - he_\Omega)he_\Omega P\\ &\quad + 2\int_\Omega (u_\Omega - he_\Omega)\, dh \wedge {}^*de_\Omega\\ &= -\int_\Omega (u_\Omega - he_\Omega)^2 P + 2\int_\Omega (u_\Omega - he_\Omega)\, dh \wedge {}^*de_\Omega\\ &\leqq 2\int_\Omega dh \wedge {}^*(d(e_\Omega u_\Omega) - e_\Omega\, du_\Omega)\\ &\quad -2\int_\Omega e_\Omega\, dh \wedge {}^*(d(he_\Omega) - e_\Omega\, dh).\end{aligned}$$

By Schwarz's inequality,

$$\begin{aligned}\|u_\Omega - he_\Omega\|_\Omega^2 \leqq\; & 2\,\|h\|_\Omega\,\|e_\Omega u_\Omega\|_\Omega + 2\,\|h\|_\Omega\,\|u_\Omega\|_\Omega\\ & + 2\,\|h\|_\Omega\,\|he_\Omega\|_\Omega + 2\,\|h\|_\Omega^2.\end{aligned}$$

By the same estimate as in §4, we deduce $\|e_\Omega u_\Omega\|_\Omega \leqq 5\|u_\Omega\|_\Omega$ and $\|u_\Omega\|_\Omega \leqq \|u_\Omega - he_\Omega\|_\Omega + \|he_\Omega\|_\Omega$, and hence

$$\|u_\Omega - he_\Omega\|_\Omega^2 \leqq 12\,\|h\|_\Omega\,\|u - he_\Omega\|_\Omega + 14\,\|h\|_\Omega\,\|he_\Omega\|_\Omega + 2\,\|h\|_\Omega^2.$$

By (5) we conclude that

$$D_\Omega(u_\Omega) \leqq K < \infty \tag{6}$$

for every Ω with a constant K.

7. Fix an Ω_0 such that $P \not\equiv 0$ on Ω_0. Since $\mathrm{PD}(\Omega_0)$ is a Hilbert space with reproducing kernel (cf. [**2**]), (6) implies that there exists an exhaustion $\{\Omega_n\}$ of R with $\Omega_n \supset \Omega_0$ such that $\{u_{\Omega_n}\}$ converges uniformly on each compact set of Ω_0. By a diagonal process, we may assume that $\{u_{\Omega_n}\}$ converges uniformly on each compact set of R. Let $u = \lim_{n\to\infty} u_{\Omega_n}$. Because of (6) and Fatou's lemma we see that $u \in \mathrm{PD}(R)$. We can regard $h - u_{\Omega_n}$ as an element of $\tilde{M}_0(R) \subset \tilde{M}_\Delta(R)$. Since $\lim_{n\to\infty}(h - u_{\Omega_n}) = h - u$ uniformly on each compact set of R and $\sup_n D_R(h - u_{\Omega_n}) < \infty$, Kawamura's lemma (cf. [7]) implies that $h - u \in \tilde{M}_\Delta(R)$, i.e. $h = Tu$ and a fortiori $h \in X_D(R)$.

References

1. M. Glasner and R. Katz, *On the behavior of solutions of $\Delta u = Pu$ at the Royden boundary*, J. Analyse Math. **22** (1969), 343–354. MR **41** #1995.

2. M. Nakai, *The space of Dirichlet-finite solutions of the equation $\Delta u = Pu$ on a Riemann surface*, Nagoya Math. J. **18** (1961), 111–131. MR **23** #A1027.

3. ———, *Dirichlet finite solutions of $\Delta u = Pu$, and classification of Riemann surfaces*, Bull. Amer. Math. Soc. **77** (1971), 381–385.

4. ———, *On the old theorem of Brelot*. Note 2: *On classification with respect to $\Delta u = Pu$*, Lecture Note at the University of California, Los Angeles, 1970.

5. M. Ozawa, *Classification of Riemann surfaces*, Kōdai Math. Sem. Rep. **1952,** 63–76. MR **14,** 462.

6. H. L. Royden, *The equation $\Delta u = Pu$, and the classification of open Riemann surfaces*, Ann. Acad. Sci. Fenn. Ser. A I No. 271 (1959). MR **22** #12215.

7. L. Sario and M. Nakai, *Classification theory of Riemann surfaces*, Die Grundlehren der math. Wissenschaften, Band 164, Springer-Verlag, New York, 1970. MR **41** #8660.

Department of Mathematics, University of California, Los Angeles, California 90024

Current address: Department of Mathematics, University of Miami, Coral Gables, Florida 33124