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SOME FIXED POINT RESULTS FOR UV DECOMPOSITIONS OF COMPACT METRIC SPACES

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ABSTRACT. In this paper the preservation of the fixed point property under UV decompositions is studied. It is shown that if K is an n-dimensional complex with the fixed point property and G is UV^{n-1} decomposition of K, then K/G also will have the fixed point property. Furthermore, if X is a compact metric space with the fixed point property, and G is a UV^n decomposition of X such that X/G may be embedded in a suitably small Euclidian space, R^m, then X/G retains the fixed point property.

Introduction. A space X is said to have the fixed point property if each continuous function $f: X \to X$ leaves some point of X fixed, i.e., there is a point $x \in X$ such that $f(x)=x$. In this paper it is shown that the fixed point property for compact metric spaces is preserved under suitable UV decompositions. Examples and some applications of the main results are described. Proofs rely to a great extent on techniques utilized by Armentrout and Price in **[1]**, and it would be helpful if the reader is familiar with this paper.

Notation and terminology. If G is an upper semicontinuous decomposition of a topological space X, then X/G will denote the associated decomposition space, and $P: X \to X/G$ the natural projection from X onto X/G.

Suppose X is a topological space, M is a subset of X, and n is a nonnegative integer. M has *property n-UV* if and only if for each open set U containing M, there is an open set V containing M such that (1) $V \subset U$ and (2) each singular n-sphere in V is homotopic to 0 in U. M has *property UV^n* if and only if for each nonnegative integer such that $i \leq n$, M has property i-UV; M has *property UV^ω* if for each nonnegative integer k, M has property k-UV.

If X is a topological space and n is a nonnegative integer, the statement that G is *UV^n decomposition* of X means that G is an upper semicontinuous decomposition of X into compact sets, each with property UV^n.

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Suppose X is a topological space. If $\mathscr{U}$ is a collection of subsets of X and $A \subset X$, then the *star* of A with respect to $\mathscr{U}$, denoted by $\mathrm{st}(A, \mathscr{U})$, is $\bigcup \{U : U \in \mathscr{U}$ and U intersects $A\}$. Suppose $\mathscr{U}$ and $\mathscr{V}$ are collections of open subsets of X. Then $\mathscr{V}$ *star refines* $\mathscr{U}$ if and only if for each set V of $\mathscr{V}$, there is a set U of $\mathscr{U}$ such that $\mathrm{st}(V, \mathscr{V}) \subset U$. If n is any nonnegative integer, then $\mathscr{V}$ *star n-homotopy refines* $\mathscr{U}$ if and only if for each set V of $\mathscr{V}$, there is a set U of $\mathscr{U}$ such that (1) $\mathrm{st}(V, \mathscr{V}) \subset U$ and (2) if $0 \leqq k \leqq n$, each singular k-sphere in $\mathrm{st}(V, \mathscr{V})$ is homotopic to 0 in U.

THEOREM 1. *Suppose K is an n-dimensional finite simplicial complex with the fixed point property and G is a UV^{n-1} upper semicontinuous decomposition of K. Then K/G has the fixed point property.*

PROOF. Suppose h is a map from K/G into itself. To find a fixed point for h, we first show that for each $\varepsilon > 0$, there exists a map $F: K \rightarrow K$ such that $d(PF, hP) < \varepsilon$, that is, $d(PF(y), hP(y)) < \varepsilon$ for each y in K/G where d is the metric of K/G. In order to apply the "lifting extension" of Price and Armentrout [**1**], we select an arbitrary point x in K, and choose $x' \in P^{-1}[hP(x)]$. Let $f: \{x\} \rightarrow \{x'\}$. Since G is a UV^{n-1} decomposition of the complex K, there exists by [**1**, Lemma 3.2, p. 435] an extension F of f to all of K with the desired property, $d(PF, hP) < \varepsilon$. Thus, for each positive integer n, we may find a map $F_n: K \rightarrow K$ such that $d(PF_n, hP) < 1/n$. Let x_n be a fixed point of F_n. Passing to a subsequence if necessary we may assume that the sequence $\{x_n\}$ converges to a point z. Then $P(z)$ will be a fixed point of h since

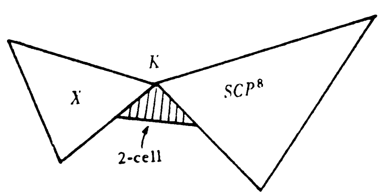
$$d(hP(z), P(z)) \leqq d(hP(z), hP(x_n)) + d(hP(x_n), PF_n(x_n)) + d(PF_n(x_n), P(z))$$

and the sum on the right side tends towards 0 for increasing values of n. Since h was arbitrary it follows that K/G has the fixed point property.

That G cannot be an arbitrary decomposition of K is seen from the following easy example.

EXAMPLE 1. Let K be the unit disk and g the unit circle in K. Then K has the fixed point property while K/G does not, where K/G is the decomposition space obtained by identifying g to a point.

A fairly natural question would be the following. Suppose G is a UV^{ω}-decomposition of a finite simplicial complex K such that K/G has the fixed point property. Does this imply that K also enjoys this property? The answer is negative as can be seen from the next example.



EXAMPLE 2. Let K be the space described in Fig. 1(b) of [3, p. 21]. Here X is the Lopez space $CP^2 \cup S_1 \times S_2 \cup CP^4$ where S_1 and S_2 are 2-spheres with S_1 identified with $CP^1 \subset CP^2$ and S_2 identified with $CP^1 \subset CP^4$. SCP^8 is the suspension of complex projective 8-space. To the wedge of X and SCP^8 a 2-cell is attached. As indicated in [3], K does not have the fixed point property. However, if we decompose the 2-cell into straight line segments (see above drawing), then the decomposition of K whose nondegenerate elements consist of precisely these segments yields a decomposition space which is the wedge of X and SCP^8 and, hence, has the fixed point property [3]. This decomposition is easily seen to be UV^ω.

If certain dimension restrictions are placed on the decomposition space, then Theorem 1 may be extended to arbitrary compact metric spaces. We first establish a lemma (Lemma 2) which is itself of some general interest.

LEMMA 1 (ARMENTROUT AND PRICE [1]). *Suppose X is a metric space, n is a nonnegative integer, G is a UV^n decomposition of X, and A is a subset of X/G. If $\mathscr{U}$ is an open covering of A, there exists an open covering $\mathscr{V}$ of A such that $\{P^{-1}[V]: V \in \mathscr{V}\}$ star n-homotopy refines $\{P^{-1}[U]: U \in \mathscr{U}\}$.*

LEMMA 2. *Suppose X is a metric space, m and n are positive integers, G is a UV^n decomposition of X, and X/G is a subset of (may be embedded in) R^m where $m \leqq n+1$. Let ε be a positive number. Then there exists an open set U of R^m which contains X/G and a map $F: U \to X$ such that $d(PF(u), u) < \varepsilon$ for each $u \in U$.*

PROOF. Let $\mathscr{W}$ be a finite open covering of X/G by sets of diameter less than $\varepsilon/2$. By repeated applications of Lemma 1, we may obtain a sequence, $\mathscr{V}^0, \mathscr{V}^1, \cdots, \mathscr{V}^m$ of finite open coverings of X/G such that $\{P^{-1}[V^0]: V^0 \in \mathscr{V}^0\}$ star n-homotopy refines $\{P^{-1}[W]: W \in \mathscr{W}\}$, and for $1 \leqq i \leqq m$,

$$\{P^{-1}[V^i]: V^i \in \mathscr{V}^i\}$$

star n-homotopy refines $\{P^{-1}[V^{i-1}]: V^{i-1} \in \mathscr{V}^{i-1}\}$. Let δ be a Lebesgue number for $\mathscr{V}^m$ and T be a triangulation of R^m with mesh less than $\min\{\delta/4, \varepsilon/2\}$. Let $N = \{\sigma \in T: \sigma$ is an m-simplex and $\sigma \cap X/G \neq \varnothing\}$. For each $\sigma \in N$, choose a point $z_\sigma \in \sigma \cap X/G$, and with each vertex v of a simplex in N, associate a point y_v in X/G in the following manner. If $v \in X/G$, then $y_v = v$, and if $v \notin X/G$, then y_v is to be a point in X/G of minimum distance from v. Note then that for every $\sigma \in N$, $\operatorname{diam}\{\{y_v: v$ a vertex of $\sigma\} \cup \{z_\sigma\}\} < \delta$. We now construct a function F from $N^* = \bigcup \{\sigma: \sigma \in N\}$ into X by first defining F on the vertices, then extending F to the 1-simplices, then to the 2-simplices, etc. If v is a vertex of a simplex in N, we let $F(v) \in P^{-1}(y_v)$. Suppose τ is a 1-simplex in some simplex of N with vertices v_1 and v_2. It follows from our construction that there is a $V^m \in \mathscr{V}^m$ such that y_{v_1} and y_{v_2} are both contained in V^m (since $d(x_{v_1}, y_{v_2}) < \delta$). Hence, $F(v_1)$ and $F(v_2)$

belong to $P^{-1}[V^m]$. Since $\{P^{-1}[V^m]: V^m \in \mathcal{V}^m\}$ is a star n-homotopy refinement of $\{P^{-1}[V^{m-1}]: V^{m-1} \in \mathcal{V}^{m-1}\}$ there is a $V^{m-1} \in \mathcal{V}^{m-1}$ such that F may be extended to a map of τ into $P^{-1}[V^{m-1}]$. Continuing in this manner (as described in [**1**, Lemma 3.2]) we may extend F to all of N^*; furthermore, F will have the property that for each $\sigma \in N$, there is a $V^0 \in \mathcal{V}^0$ such that $F[\sigma] \subset P^{-1}[V^0]$. Observe also that V^0 may be chosen so that $z_\sigma \in V^0$, and of course, $V^0 \subset W$ for some $W \in \mathcal{W}$.

Let U=Interior N^*. To complete the proof we need to check that if $u \in U$, then $d(PF(u), u) < \varepsilon$. For $u \in U$ there exists a $\sigma \in N$ such that $u \in \sigma$. Then it is easily verified that

$$d(PF(u), u) \leqq d(PF(u), z_\sigma) + d(z_\sigma, u) < \varepsilon/2 + \varepsilon/2$$

since $PF(u)$ and z_σ both belong to some $W \in \mathcal{W}$.

THEOREM 2. *Suppose G is a UV^n decomposition of a compact metric space X such that X/G may be embedded in R^m for some $m \leqq n+1$. Then if X has the fixed point property, so does X/G.*

PROOF. Suppose that X/G does not have the fixed point property. Let $K: X/G \to X/G$ be a map which is fixed point free. Since X/G is compact there is a positive number ε such that $d(K(y), y) \geqq \varepsilon$ for each $y \in X/G$. By Lemma 2, we may find an open set U in R^m which contains X/G and a map $F: U \to X$ such that $d(PF(u), u) < \varepsilon$ for each $u \in U$. Define $g = F \circ K \circ P$. Thus g is a map from X into X and, hence, has a fixed point z. But $d(KP(z), P(z)) < \varepsilon$ since $P(z) = PF(KP(z))$ and $d(PF(KP(z)), KP(z)) < \varepsilon$. Since this contradicts the fact that $d(K(y), y) \geqq \varepsilon$ for each $y \in X/G$, K must have a fixed point.

COROLLARY 1. *Suppose G is a UV^ω decomposition of a compact metric space X such that X/G is finite dimensional. Then if X has the fixed point property, so does X/G.*

COROLLARY 2. *Suppose G is a UV^n decomposition of a compact metric space X such that* dim $X/G \leqq n/2$. *Then if X has the fixed point property, so does X/G.*

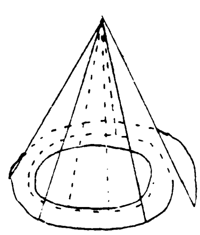


FIGURE 1

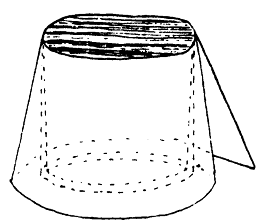


FIGURE 2

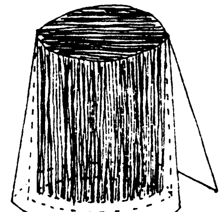


FIGURE 3

PROOF. Since $2(n/2)+1 \leq n+1$, X/G may be embedded in R^{n+1} and Theorem 2 applies.

As an application of Theorem 2, we consider the following spaces. Let X be the cone over a circle with convergent spiral (Figure 1). Let Y be the bottomless-can-with-skirt-with-lid-attached (Figure 2; see [**2**, p. 131]), and let Z be the solid can-with-skirt (Figure 3).

X does not have the fixed point property (see, for example, [**2**, p. 129]), and, hence, Y cannot have the fixed point property; for if it did, the decomposition space obtained by identifying the lid to a point (a UV^ω decomposition) would also have the fixed point property—but this space is just X. It then follows that Z fails to have the fixed point property (see also Knill [**4**]) since Z retracts onto Y.

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