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A NOTE ON LÉVY'S BROWNIAN PROCESS ON THE HILBERT SPHERE

PEGGY TANG STRAIT[1]

ABSTRACT. Let $X(t)$, t in l_2, be Lévy's separable Brownian process. Let S be the unit Hilbert sphere. It is shown that with probability 1, the image of $X(t)$, $t\in S$, is the entire real line.

1. **Introduction.** Let $X(t)$, t in l_2, be Lévy's separable Brownian process with $X(0)=0$, mean 0, and covariance function

$$r(x, t) = E(X(s)X(t)) = (1/2)(\|s\| + \|t\| - \|t - s\|).$$

Let S be the unit Hilbert sphere $S=\{t\in l_2: \|t\|\leqq 1\}$.

THEOREM. *With probability* 1, *the image of* $X(t)$, $t\in S$, *is the entire real line.*

2. **Preliminary lemmas.** Let $A_n=(i_1, i_2, \cdots)$ be points in S defined by $i_n=1$, $i_j=0$ if $j\neq n$. Let $X_n=X(A_n)$. Lévy proved the following lemma.

LEMMA 1 [**1**]. *The sequence* $\{X_n\}$ *is almost surely not bounded.*

LEMMA 2 [**2**]. *Let T be a subset of l_2 of the form* $T=\{(t_1, t_2, \cdots): a_n\leqq t_n\leqq a_n+1/2^n, (a_1, a_2, \cdots)\in l_2\}$. *With probability* 1, $X(t)$ *is continuous for t in T.*

The proof of Lemma 2 is given in [**2**].

3. **Proof of the Theorem.** In terms of the random variables X_n, $n=1, 2, \cdots$, of §2 above, define random variables Y_i, $i=1, 2, \cdots$,

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successively by the formulas

$$
(3.1)\qquad
\begin{aligned}
X_1 &= Y_1, \\
X_2 &= a_{2,1}Y_1 + Y_2, \\
&\cdot \\
&\cdot \\
&\cdot \\
X_n &= a_{n,1}Y_1 + a_{n,2}Y_2 + \cdots + a_{n,n-1}Y_{n-1} + Y_n, \\
&= \sum_{i=1}^{n} a_{n,i}Y_i,
\end{aligned}
$$

with $a_{n,n}=1$ and $a_{i,j}$ chosen so that $E(Y_iY_j)=0$ whenever $i\neq j$. Then $\{Y_i\}$ is a sequence of independent normal random variables each with mean zero. We can therefore apply Kolmogorov's zero-one law and conclude that the tail event "for every $c>0$ and $n>0$ there is an $m>0$ such that $\sum_{i=1}^{m} a_{m,i}Y_i>c$" has for probability either 0 or 1. Similarly the tail event "for every $c>0$ and $n>0$ there is an $m>0$ such that $\sum_{i=1}^{m} a_{m,i}Y_i<-c$" has for probability either 0 or 1. By symmetry of the random variables Y_i, the probability of the two tail events described above must be equal, hence either both equal to 0 or both equal to 1. Now apply Lemma 1 and we conclude that the probability is 1, or in other words, with probability one the sequence $\{X_n\}$ takes on arbitrarily large and arbitrarily small (i.e. very negative) values.

Let $x(t)$ denote sample functions of the process $X(t)$ and x_n denote values of X_n (i.e. $x(A_n)$). Let B be the set of all sample functions $x(t)$ such that the sequence $\{x_n\}$ takes on arbitrarily large and arbitrarily small values. We showed in the above paragraph that

$$
(3.2)\qquad P(B) = 1.
$$

For each n, $n=1, 2, \cdots$, define

$$
(3.3)\qquad T_n = \{(t_1, t_2, \cdots):0 \leqq t_n \leqq 1, t_k = 0 \text{ if } k \neq n\}.
$$

Observe that T_n is a subset of S. Let C_n be the set of all sample functions $x(t)$ for which $x(t)$ is continuous for t in T_n. Since T_n is covered by the union of a finite number of sets of the form T in Lemma 2, we have

$$
(3.4)\qquad P(C_n) = 1, \qquad n = 1, 2, \cdots.
$$

Let $D=B\cap(\bigcap_{n=1}^{\infty} C_n)$. It follows from equations (3.2) and (3.4) that

$$
(3.5)\qquad P(D) = 1.
$$

Let $x(t)$ be a sample function in D. Let c be an arbitrary positive real number. Since $x(t)$ is also in B, there is a point A_n (as defined in §2) and a point A_m such that

$$x_n = x(A_n) > c \quad \text{and} \quad x_m = x(A_m) < -c. \tag{3.6}$$

Now, A_n is a point in the set T_n and $0=(0, 0, \cdots)$ is also a point in T_n. Furthermore, there is a continuous path in T_n joining the point A_n with the point 0. Since $x(t)$ is in D and thus also in C_n, $x(t)$ is continuous along this path. Hence $x(t)$ must take on all values between $x(0)=0$ and $x_n=x(A_n)$. Thus, there exists a point t in T_n, and therefore also in S, such that $x(t)=c$. Similarly, there is a point s in S such that $x(s)=-c$.

4. **Remark.** The Hilbert sphere S in the theorem may be replaced by any noncompact subset T of l_2 of the form $T=\{(t_1, t_2, \cdots): a_n \leqq t_n \leqq b_n\}$ by applying the statement proved in [2] that almost all sample functions $x(t)$ are unbounded in T.

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DEPARTMENT OF MATHEMATICS, QUEENS COLLEGE OF THE CITY UNIVERSITY OF NEW YORK, FLUSHING, NEW YORK 11367