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ON AN IDENTITY OF ECKFORD COHEN[1]

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ABSTRACT. We characterize all multiplicative arithmetical functions $f_k(r)$ such that an identity of the form

$$\sum_{r=1}^{\infty} f_k(r)c_k(n, r) = q_k(n)g(k), \qquad g(k) \neq 0,$$

holds for all n, where $q_k(n)$ is the characteristic function of the set of k-free integers and $c_k(n, r)$ is the generalized Ramanujan sum. This characterization yields several arithmetical identities of the above form including an identity of Eckford Cohen, which occurs as a special case of our theorem on taking $f_k(r)=\mu(r)/J_k(r)$ and $g(k)=\zeta(k)$.

1. **Introduction.** Throughout the following k denotes a fixed integer $\geqq 2$. Let Q_k denote the set of k-free integers, that is, the integers whose prime factors are all of multiplicity $<k$. Let $q_k(n)=1$ or 0 according as $n \in Q_k$ or $n \notin Q_k$. Also, let $\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$ for $s>1$.

In 1963, E. Cohen [3, (1)] established the following identity and used it to obtain an elementary estimate for $Q_k(x)$, the number of k-free integers $\leqq x$:

$$\sum_{r=1}^{\infty} \left(\frac{\mu(r)}{J_k(r)}\right) c_k(n, r) = q_k(n)\zeta(k), \tag{1}$$

where $\mu(r)$ is the Möbius function, $J_k(r)$ the Jordan totient function of order k, and $c_k(n, r)$ is the generalized Ramanujan sum defined by

$$c_k(n, r) = \sum_{a(\mathrm{mod}\, r^k);(a,r^k)_k=1} \exp\left(\frac{2\pi i a n}{r^k}\right), \tag{2}$$

the summation being over all $a(\mathrm{mod}\, r^k)$, whose greatest common kth power divisor with r^k is 1. This generalization of Ramanujan's sum was introduced under a slightly different notation by E. Cohen [2] himself in

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1949, who also established [**2**, (2.5)] the following arithmetic evaluation of $c_k(n, r)$:

$$c_k(n, r) = \sum_{d|r;\, d^k|n} d^k \mu(r/d). \tag{3}$$

The object of the present paper is to characterize all multiplicative arithmetical functions $f_k(r)$ such that an identity of the following type holds for all n:

$$\sum_{r=1}^{\infty} f_k(r) c_k(n, r) = q_k(n) g(k), \qquad g(k) \neq 0. \tag{4}$$

It may be noted that (1) is a particular case of (4) with $f_k(r) = \mu(r)/J_k(r)$ and $g(k) = \zeta(k)$.

The characterization of $f_k(r)$ satisfying (4) is given by the following:

THEOREM. *Let $f_k(r)$ be a multiplicative arithmetic function such that $\sum_{r=1}^{\infty} f_k(r)c_k(n, r)$ is absolutely convergent for all n. Then the identity* (4) *holds if and only if*

(i) $f_k(p^\alpha) = (1 + f_k(p)(p^k - 1))/p^k$ *for every* $\alpha \geqq 2$ *and for every prime* p;

(ii) $g(k) \neq 0$ *and is given by* $g(k) = \prod_p \{1 - f_k(p)\}$, *where the product is extended over all primes* p.

REMARK. For the function $f_k(r) = \mu(r)/J_k(r)$, we note that $f_k(p) = -1/(p^k - 1)$ and $f_k(p^\alpha) = 0$ for every $\alpha \geqq 2$, since $J_k(r) = r^k \prod_{p|r} (1 - 1/p^k)$; so that the conditions of the theorem are satisfied, thus yielding the identity (1).

2. **Proof of the theorem.** Since $f_k(r)$ and $c_k(n, r)$ are multiplicative functions of r and the series $\sum_{r=1}^{\infty} f_k(r)c_k(n, r)$ is absolutely convergent, it can be expanded into an infinite product of Euler type [**4**, Theorem 286]. Hence by (3), we have

$$\begin{aligned} \sum_{r=1}^{\infty} f_k(r) c_k(n, r) &= \prod_p \left\{ \sum_{m=0}^{\infty} f_k(p^m) c_k(n, p^m) \right\} \\ &= \prod_p \left\{ \sum_{m=0}^{\infty} f_k(p^m) \sum_{d|p^m;\, d^k|n} d^k \mu\left(\frac{p^m}{d}\right) \right\}. \end{aligned} \tag{5}$$

First suppose that $f_k(p^\alpha) = (1 + f_k(p)(p^k - 1))/p^k$, for every $\alpha \geqq 2$ and for every prime p, and $g(k) \neq 0$ which is given by $g(k) = \prod_p \{1 - f_k(p)\}$. Then we prove the identity (4), which is equivalent to proving

$$\begin{aligned} \sum_{r=1}^{\infty} f_k(r) c_k(n, r) &= \prod_p \{1 - f_k(p)\}, && \text{if } n \in Q_k, \\ &= 0, && \text{if } n \notin Q_k. \end{aligned} \tag{6}$$

Now, if $n \in Q_k$, then it is clear that 1 is the only kth power divisor of n, so that from (5), we get that

$$\sum_{r=1}^{\infty} f_k(r)c_k(n, r) = \prod_p \left\{\sum_{m=0}^{\infty} f_k(p^m)\mu(p^m)\right\} = \prod_p \{1 - f_k(p)\}.$$

Let $n \notin Q_k$. Then n can be written uniquely in the form $n = n_1 n_2$, where $n_2 > 1$, $(n_1, n_2) = 1$, $n_1 \in Q_k$, $n_2 \in L_k$, L_k being the set of k-full integers, that is, the integers whose prime factors are all of multiplicity $\geqq k$. Let $n_2 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$. Since $n_2 \in L_k$, $\alpha_i \geqq k$ for $1 \leqq i \leqq t$. We can write $\alpha_i = u_i k + v_i$, where $u_i \geqq 1$ and $0 \leqq v_i \leqq k-1$. From (5) we get that

$$\begin{aligned}
\sum_{r=1}^{\infty} f_k(r)c_k(n, r) &= \prod_p \left\{\sum_{m=0}^{\infty} f_k(p^m) \sum_{d \mid p^m;\, d^k \mid n_1 n_2} d^k \mu\left(\frac{p^m}{d}\right)\right\} \\
&= \prod_{p;\, p \nmid n_2} \left\{\sum_{m=0}^{\infty} f_k(p^m)\mu(p^m)\right\} \cdot \prod_{p \mid n_2} \left\{\sum_{m=0}^{\infty} f_k(p^m) \sum_{d \mid p^m;\, d^k \mid n_2} d^k \mu\left(\frac{p^m}{d}\right)\right\} \\
&= \prod_{p;\, p \nmid n_2} \{1 - f_k(p)\} \cdot \prod_{i=1}^{t} \left\{\sum_{m=0}^{\infty} f_k(p^m) \sum_{d \mid p_i^m;\, d^k \mid p_i^{\alpha_i}} d^k \mu\left(\frac{p_i^m}{d}\right)\right\} \qquad (7) \\
&= \prod_{p;\, p \nmid n_2} \{1 - f_k(p)\} \\
&\quad \cdot \prod_{i=1}^{t} \left\{1 + \sum_{a=1}^{u_i} f_k(p_i^a)[p_i^{ak} - p_i^{(a-1)k}] - f_k(p_i^{u_i+1}) p_i^{u_i k}\right\}.
\end{aligned}$$

Since $f_k(p^\alpha) = (1 + f_k(p)(p^k - 1))/p^k$ for every $\alpha \geqq 2$ and for every prime p, we have

$$f_k(p_i^2) = f_k(p_i^3) = \cdots = f_k(p_i^{u_i+1}) = (1 + f_k(p_i)(p_i^k - 1))/p_i^k$$

for $1 \leqq i \leqq t$. Hence each term in the second product is zero, so that

$$\sum_{r=1}^{\infty} f_k(r)c_k(n, r) = 0 \quad \text{for all } n \notin Q_k.$$

Thus (6) is proved, so that the first part of the theorem follows.

Conversely, suppose that (4) holds for all n with $g(k) \neq 0$. Taking $n = 1$, we see from (3) that $c_k(1, r) = \mu(r)$, so that from (4) and (5), we get that

$$g(k) = \sum_{r=1}^{\infty} f_k(r)\mu(r) = \prod_p \{1 - f_k(p)\},$$

so that condition (ii) of the theorem is satisfied. Since $g(k) \neq 0$, we have $f_k(p) \neq 1$ for every prime p.

Let q be any arbitrary prime. To prove condition (i) of the theorem, it is enough if we prove that

(8) $$f_k(q^\alpha) = (1 + f_k(q)(q^k - 1))/q^k \quad \text{for every } \alpha \geqq 2.$$

We prove (8) by complete induction on α.

Now, taking $n=q^k$ in (4), we get from (7) (in this case $n_1=1$, $n_2=q^k$, $i=1$ and $u_1=1$) that

$$0 = \prod_{p;p\neq q} \{1 - f_k(p)\} \cdot \{1 + f_k(q)(q^k - 1) - f_k(q^2)q^k\}$$
$$= [g(k)/(1 - f_k(q))] \cdot \{1 + f_k(q)(q^k - 1) - f_k(q^2)q^k\}.$$

Since $g(k)\neq 0$, we get that

$$f_k(q^2) = (1 + f_k(q)(q^k - 1))/q^k,$$

that is, (8) is true for $\alpha=2$.

Suppose that (8) is true for $\alpha=2, 3, \cdots, \beta$, where $\beta\geqq 2$. We shall prove (8) for $\alpha=\beta+1$. Taking $n=q^{\beta k}$ in (4), we get from (7) (in this case $n_1=1$, $n_2=q^{\beta k}$, $i=1$ and $u_1=\beta$) that

$$0 = \prod_{p;p\neq q} \{1 - f_k(p)\} \cdot \left\{1 + \sum_{a=1}^{\beta} f_k(q^a)[q^{ak} - q^{(a-1)k}] - f_k(q^{\beta+1})q^{\beta k}\right\},$$

so that

$$\begin{aligned} q^{\beta k}f_k(q^{\beta+1}) &= 1 + f_k(q)(q^k - 1) + \sum_{a=2}^{\beta} f_k(q^a)[q^{ak} - q^{(a-1)k}] \\ &= 1 + f_k(q)(q^k - 1) + f_k(q^2)[q^{\beta k} - q^k] \\ &= 1 + f_k(q)(q^k - 1) + \left[\frac{1 + f_k(q)(q^k - 1)}{q^k}\right](q^{\beta k} - q^k) \\ &= q^{(\beta-1)k}[1 + f_k(q)(q^k - 1)], \end{aligned}$$

that is, $f_k(q^{\beta+1})=(1+f_k(q)(q^k-1))/q^k$. Hence (8) follows for every $\alpha\geqq 2$. Thus the proof of the theorem is complete.

3. **Some special cases.** Taking

$$f_k(p) = \frac{A}{p^k - B} \quad \text{and} \quad f_k(p^\alpha) = \frac{(A + 1)p^k - (A + B)}{p^k(p^k - B)}$$

for every $\alpha\geqq 2$ and for every prime p, where A and B are constants which have the values 0, 1 or -1, we see that the conditions of the theorem are satisfied with $g(k)=\prod_p \{1-A/(p^k-B)\}$. Hence we have the following identities corresponding to the values of A and B given by (1, 0), (1, -1), (1, 1), (−1, 0), (−1, −1), (−1, 1):

(9) $$\sum_{r=1}^{\infty} f_k(r)c_k(n, r) = \frac{q_k(n)}{\zeta(k)},$$

(10) $$\sum_{r=1}^{\infty} f_k(r)c_k(n, r) = q_k(n)\frac{\zeta(2k)}{\zeta(k)},$$

(11) $$\sum_{r=1}^{\infty} f_k(r)c_k(n, r) = q_k(n)\zeta(k)\prod_p \left(1 - \frac{2}{p^k}\right),$$

(12) $$\sum_{r=1}^{\infty} f_k(r)c_k(n, r) = q_k(n)\frac{\zeta(k)}{\zeta(2k)},$$

(13) $$\sum_{r=1}^{\infty} f_k(r)c_k(n, r) = q_k(n)\frac{\zeta(2k)}{\zeta(k)}\prod_p \left(1 + \frac{2}{p^k}\right),$$

(14) $$\sum_{r=1}^{\infty} f_k(r)c_k(n, r) = q_k(n)\zeta(k).$$

The infinite products in the right sides of (11) and (13) can be expressed as infinite products involving the Riemann zeta function using the following result.

$$\prod_p \left(1 - \frac{\alpha}{p^s}\right)^{-1} = \prod_{m=1}^{\infty} (\zeta(ms))^{a(m)},$$

for every α and $s>1$, where $a(m)=m^{-1} \sum_{d|m} \alpha^d \mu(m/d)$.

For a proof of this result, we refer to [1].

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