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A CHARACTERIZATION OF GROUPS WITH ISOMORPHIC SUBGROUPS

DONALD W. MILLER

ABSTRACT. The concept of normalizer is generalized to derive a characterization of groups G which contain a proper subgroup isomorphic to G.

The problem of what groups have proper isomorphic subgroups has attracted the attention of a number of authors. Baer **[1]** obtained a characterization of groups without proper isomorphic subgroups, while Beaumont **[2]**, Kaplansky **[4]** and Clay **[3]** identified a number of classes of groups with proper isomorphic subgroups as well as several classes of groups without such subgroups. It is the purpose of this note to give an explicit characterization of groups with proper isomorphic subgroups.[1]

Let H be a subgroup of the group G, with normalizer $N=N_G(H)$, where $N=\{x\in G|xH=Hx\}$. Define the *left normalizer*, $L=L_G(H)$, of H in G by $L=\{x\in G|xH\subseteq Hx\}$ and the *right normalizer*, $R=R_G(H)$, of H in G by $R=\{x\in G|Hx\subseteq xH\}$. The following observations are immediate consequences of these definitions.

LEMMA. *Let H be a subgroup of a group G. Then*:
(i) $L\cap R=N$;
(ii) $R=L^{-1}=\{x^{-1}|x\in L\}$;
(iii) *either $L=R=N$, or L, R and N are pairwise distinct.*

The criterion referred to in the title can now be established.

THEOREM. *A group H has an isomorphic proper subgroup if and only if H is imbeddable in a group G with the property that the left and right normalizers of H in G are distinct.*

PROOF. Suppose that H is imbeddable in a group G such that $L=L_G(H)\neq R=R_G(H)$. Then $L\supset N=N_G(H)$ so, selecting a in $L\backslash N$ (the complement of N in L), we have $aH\subset Ha$. Thus, aHa^{-1} is a proper subgroup of H which is isomorphic to H.

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[1] The generalizations of the normalizer of a subgroup on which this characterization is based were suggested to the writer by Thomas Shores.

Conversely, let H_1 be a group which contains a proper isomorphic subgroup H_0, with φ_0 an isomorphism of H_0 onto H_1. Then

$$H_0\varphi_0 = H_1 \supset H_0$$

so there exists a proper subgroup $H_{-1}=H_0\varphi_0^{-1}$ of H_0 such that $H_{-1}\varphi_0=H_0$. Consequently $\varphi_{-1}=\varphi_0|H_{-1}$ (the restriction of φ_0 to H_{-1}) is an isomorphism of H_{-1} onto H_0. Iteration of this process leads to a strictly decreasing chain of subgroups $H_1\supset H_0\supset H_{-1}\supset\cdots$ of H_1, together with a sequence of surjective isomorphisms $\varphi_i: H_i \twoheadrightarrow H_{i+1}$, $i=0, -1, -2, \cdots$, such that

$$\varphi_i \mid H_{i-1} = \varphi_{i-1}, \qquad i = 0, -1, -2, \cdots. \tag{1}$$

Let A_1 be a set disjoint from H_1 such that $|A_1|=|H_1\backslash H_0|$, and let β be a bijection from $H_1\backslash H_0$ to A_1. Define a bijection φ_1 from H_1 to $H_2=H_1\cup A_1$ by

$$\varphi_1 \mid H_0 = \varphi_0, \qquad \varphi_1 \mid (H_1\backslash H_0) = \beta.$$

Extend the definition of multiplication from H_1 to H_2 by stipulating that

$$xy = ((x\varphi_1^{-1})(y\varphi_1^{-1}))\varphi_1, \quad \text{all } x, y \in H_2,$$

(which is clearly consistent with the existing multiplication in H_1). Then for all $x, y \in H_1$,

$$(x\varphi_1)(y\varphi_1) = ((x\varphi_1\varphi_1^{-1})(y\varphi_1\varphi_1^{-1}))\varphi_1 = (xy)\varphi_1,$$

so φ_1 is an isomorphism of H_1 onto H_2.

Repetition of this procedure yields a strictly increasing chain of groups $H_1\subset H_2\subset H_3\subset\cdots$, together with a sequence of isomorphisms φ_i: $H_i\twoheadrightarrow H_{i+1}$, $i=1, 2, 3, \cdots$, such that

$$\varphi_i \mid H_{i-1} = \varphi_{i-1}, \qquad i = 1, 2, 3, \cdots. \tag{2}$$

Now let K be the group given by $K=\bigcup_{i=-\infty}^{\infty} H_i$ and define the transformation φ of K by $\varphi: x\mapsto x\varphi_i$, all $x\in H_i$; by (1) and (2), φ is well defined. It follows that φ is an automorphism of K. Thus, φ is induced by an inner automorphism, say $T(\alpha)$, of the holomorph G of K. Consequently,

$$\alpha^{-1}H_1\alpha = H_1T(\alpha) = H_1\varphi = H_1\varphi_1 = H_2 \supset H_1,$$

so $H_1\alpha\supset\alpha H_1$. Therefore $\alpha\in L_G(H_1)\backslash N_G(H_1)$, so, by the lemma, $L_G(H_1)\neq R_G(H_1)$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA, LINCOLN, NEBRASKA 68508