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A NECESSARY AND SUFFICIENT CONDITION FOR $\beta X\backslash X$ TO BE AN INDECOMPOSABLE CONTINUUM

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ABSTRACT. In his dissertation, David Bellamy has shown that if $I=[0, 1)$, then $\beta I\backslash I$ is an indecomposable continuum, and R. G. Woods, in his dissertation, obtained the same result and in addition showed that for $m>1$, $\beta R^m\backslash R^m$ is a decomposable continuum. In this note we give a necessary and sufficient condition for $\beta X\backslash X$ to be an indecomposable continuum when X is a locally connected generalized continuum.

Definitions and notation. If Y is a topological space and $A\subset B\subset Y$, we denote the closure of A in B by $\mathrm{cl}_B A$ and we say that a set $A\subset B$ is *conditionally compact* in B if $\mathrm{cl}_B A$ is compact. A *ray* in Y is a closed subset R of Y that is homeomorphic to $[0, 1)$. We say that a connected space Y has the *strong complementation* property provided whenever U is a non-conditionally compact connected open subset of Y, $Y\backslash U$ is compact. A compact connected Hausdorff space Y is said to be an *indecomposable continuum* if there does not exist two nonempty closed, connected proper subsets of Y whose union is Y. By a *generalized continuum* we will mean a locally compact, connected metric space. We will use βX to denote the Stone-Čech compactification of a completely regular Hausdorff space. All terms not defined here may be found in [**6**], [**7**] or [**8**].

We need the following results:

LEMMA 1 [**7**, THEOREM 3.41]. *Let Y be a compact, connected Hausdorff space and C be a nonempty, closed and connected proper subset of Y. If C has a nonempty interior, Y is not an indecomposable continuum.*

LEMMA 2. *Let X be a noncompact, locally connected generalized continuum. Then X contains a ray.*

PROOF. Let X_∞ denote the one-point compactification of X. It is well known that X_∞ is a locally connected, compact connected metric space and as such is arcwise connected. Let $a\in X$ and let I be any arc in X_∞ from a to $\{\infty\}$. Then $I\backslash\{\infty\}$ is a ray in X.

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LEMMA 3. *Let X be a noncompact, locally connected generalized continuum. Then X has the strong complementation property if and only if for every ray R in X, $X\setminus R$ is conditionally compact.*

PROOF OF THE NECESSITY. Let R be a ray in X and suppose that $X\setminus R$ is not conditionally compact. Then there exists a closed set A of X such that A is not compact and $A\cap R=\varnothing$. Since X is locally connected there exists an open connected set U of X containing R such that $A\cap U=\varnothing$. But then neither $\mathrm{cl}_X U$ or $X\setminus U$ is compact and so X cannot have the strong complementation property. This completes the proof of the necessity.

PROOF OF THE SUFFICIENCY. Suppose that for every ray R in X, $X\setminus R$ is conditionally compact. Let U be any nonconditionally compact open connected subset of X. By Lemma 2, X contains a ray and so there exists a homeomorphism of h of $[0, 1)$ onto a closed set S of X. Since $\mathrm{cl}_X(X\setminus S)$ is compact there exists a point $t\in[0, 1)$ such that $h(t)$ is the last point on S that lies in $\mathrm{cl}_X(X\setminus S)$. Then $h(t, 1)$ is an open subset of X. We assert that there exists $s\in[t, 1)$ such that $h[s, 1)\subset U$. First we note that for every $s\in[t, 1)$, $h[s, 1)\cap U\neq\varnothing$ for otherwise, since $X\setminus h[s, 1)$ is conditionally compact, U would be conditionally compact. Now suppose that for every $s\in[t, 1)$, $h[s, 1)$ is not a subset of U. Then there exists $p\in(t, 1)$ such that $h(p)\notin U$ and both of the sets $h[t, p)$ and $h(p, 1)$ intersect U. Now $U_1=U\cap(X\setminus h[p, 1))$ is open in X since $h[p, 1)$ is closed and $U_2=U\cap h(p, 1)$ is open in X since $h(p, 1)$ is open in X. Furthermore, neither U_1 or U_2 is empty and $U=U_1\cup U_2$. Of course, this contradicts the assumption that U is connected and so there exists $s\in[t, 1)$ such that $h[s, 1)\subset U$. Then $X\setminus U$ is a subset of $X\setminus h[s, 1)$ and so $X\setminus U$ is compact. This completes the proof.

THEOREM. *Let X be a noncompact locally connected generalized continuum. Then a necessary and sufficient condition for $\beta X\setminus X$ to be an indecomposable continuum is that X have the strong complementation property.*

PROOF OF THE SUFFICIENCY. Suppose that X has the strong complementation property. Let R be any ray in X and let $T=\mathrm{cl}_{\beta X}R$. By D. Bellamy's result, $\beta R\setminus R$ is an indecomposable continuum and by Theorem 7 of [4], the identity mapping on R can be extended to a homeomorphism of T onto βR. Thus, $C=T\setminus R$ is an indecomposable continuum that lies in $\beta X\setminus X$. But by Lemma 3, $\mathrm{cl}_X(X\setminus R)$ is compact so that $\beta X\setminus X=C$ and $\beta X\setminus X$ is an indecomposable continuum.

PROOF OF THE NECESSITY. Suppose that X does not have the strong complementation property. By Lemma 2, there exists a ray R in X such that $\mathrm{cl}_X(X\setminus R)$ is not compact. Then there exists a noncompact closed subset A of X such that $A\cap R=\varnothing$ and a connected open subset U of X such that $R\subset U$ and $\mathrm{cl}_X U\cap A=\varnothing$. Now since X is locally compact and separable $X=\bigcup K_i$, $i=1, 2, \cdots$, where for each $i\geqq 1$, K_i is a compact subset of the interior of K_{i+1}. Let $R_1, R_2, \cdots$ be a sequence of subrays of R such that for each $i\geqq 1$, R_i is a subset of $X\backslash K_i$. Finally for each $i\geqq 1$ let G_i be the component of $U\cap(X\backslash K_i)$ that contains R_i and let $H_i=(X\backslash K_i)\cap(X\backslash G_i)$. We set $G=\bigcap \mathrm{cl}_{\beta X}G_i$, $i=1, 2, \cdots$, and $H=\bigcap \mathrm{cl}_{\beta X}H_i$, $i=1, 2, \cdots$, and note that $\beta X\backslash X=H\cup G$. Furthermore, we observe that G is a nonempty compact, connected space. Now if $\beta X\backslash X$ is not connected, it cannot be an indecomposable continuum so suppose that $\beta X\backslash X$ is connected. We now proceed to show that G is a proper subset of $\beta X\backslash X$ and that G has a nonempty interior. Let $a\in\mathrm{cl}_{\beta X}A\cap(\beta X\backslash X)$. Note that such a point exists since A is not compact. Then since X is a normal space, A and $\mathrm{cl}_X U$ are completely separated in X and by Theorem (6.5) of [6], A and $\mathrm{cl}_X U$ have disjoint closures in βX. Then $a\notin\mathrm{cl}_{\beta X}U\supset G$ and so $G\neq\beta X\backslash X$. Similarly if $r\in\mathrm{cl}_{\beta X}R_1\cap(\beta X\backslash X)$, $r\notin\mathrm{cl}_{\beta X}H_1\supset H$ and so G has a nonempty interior relative to $\beta X\backslash X$. By Lemma 1, $\beta X\backslash X$ is not an indecomposable continuum and this completes the proof.

COROLLARY. *Let X be a noncompact locally connected generalized continuum. If $\beta X\backslash X$ is an indecomposable continuum, then X_∞, the one-point compactification of X, is the only locally connected compactification of X.*

PROOF. Suppose that Y is a locally connected compactification of X and $Y\backslash X$ is nondegenerate. Then there exists disjoint arcs I_1 and I_2 in Y such that $I_i\cap(Y\backslash X)$ is a single point a_i, $i=1, 2$. Then if $R_i=I_i\backslash\{a_i\}$, $i=1, 2$, R_1 is a ray in X with $X\backslash R_1$ nonconditionally compact. By Lemma 3, X does not have the strong complementation property and hence $\beta X\backslash X$ cannot be an indecomposable continuum. Of course, this implies that X_∞ is the only locally connected compactification of X.

EXAMPLE. Let $T_0=[0, 1)$ and for each positive integer n let T_n be the closed line segment joining the points $(n-1/n, 1)$ and $(n-1/n, 0)$ in E^2. Let $X=\bigcup T_i$, $i=0, 1, 2, \cdots$. Then X is a locally connected generalized continuum that does not have the strong complementation and so $\beta X\backslash X$ is not indecomposable. However, X_∞ is the only locally connected compactification of X.

REMARK. A connected space Y is said to have the complementation property if for every compact set K in Y, $Y\backslash K$ has at most one nonconditionally compact component It can be shown that if X is a locally connected generalized continuum, then X has the complementation if and only if $\beta X\backslash X$ is connected. Then by the theorem above, X has the complementation property whenever it has the strong complementation property. See [2] or [3] for further properties of spaces with the complementation property.

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