PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 1, May 1972

ON SPACES WITH NORMS OF NEGATIVE AND POSITIVE ORDER[1]

GIDEON PEYSER

ABSTRACT. The two Hilbert spaces H_0 and H_1 are defined to be a generating pair if H_1 is a dense subspace of H_0 and if the norm of an element in H_1 is greater than or equal to the norm in H_0. It is shown that the pair generates a sequence of spaces $\{H_k\}$, $-\infty < k < \infty$, such that any two spaces of the sequence form again a generating pair. Such a pair is shown to generate, in turn, a subsequence of $\{H_k\}$. Also, representation theorems are derived for bounded linear functionals over the spaces of the sequence $\{H_k\}$, generalizing the Lax representation theorem and the Lax-Milgram theorem.

1. **Introduction.** We consider two Hilbert spaces, H_0 with norm $\| \ \|_0$ and inner product $(\ , \ )_0$, and H_1 with norm $\| \ \|_1$ and inner product $(\ , \ )_1$, having the following properties. H_1, as a vector space, is isomorphic to a subspace $\hat{H}_1$ of H_0; $\hat{H}_1$ is dense in H_0; if $x_0 \in \hat{H}_1$ corresponds to $x_1 \in H_1$ then $\|x_0\|_0 \leqq \|x_1\|_1$.

We shall identify the spaces H_1 and $\hat{H}_1$ but will consider their elements equipped with two sets of norms and inner products. Hence we have for $x \in H_1$ that

$$\|x\|_0 \leqq \|x\|_1. \tag{1}$$

The spaces H_0 and H_1, with the above properties, will be called a *generating pair*. H_1 is the *stronger* and H_0 the *weaker* space of the pair.

In §2 we construct from the generating pair a sequence of Hilbert spaces $\{H_k\}$, $-\infty < k < \infty$, such that any two spaces H_{k_1} and H_{k_2}, $k_1 < k_2$, form a generating pair with H_{k_1} the weaker and H_{k_2} the stronger space. In [3] Lax considers the space G of square integrable functions and the space $G^{(r)}$ of functions with square integrable derivatives up to order r. G and $G^{(r)}$ form, in our notation, a generating pair. Lax constructs the space $G^{(-r)}$,

Presented to the Society, January 19, 1972 under the title *On negative and positive norms*; received by the editors March 28, 1971.

AMS 1970 *subject classifications*. Primary 46C10; Secondary 46B10.

Key words and phrases. Hilbert space, norm, inner product, weaker space, stronger space, lower space, upper space, isometric isomorphism, correspondence, bounded linear functional, representation theorem.

[1] This paper was supported in part by the NSF grant GP-8601.

© American Mathematical Society 1972

the space of functions with norms of negative order. These spaces play an important role in existence and regularity theorems in partial differential equations. For further details regarding these negative norms, see Schechter **[6]** and Yosida [7]. In **[1]** Landesman constructs, in a generalized way, spaces that correspond to the negatively normed spaces of Lax, and analyzes their properties.

The main results are in §3 where we show that the sequence of $\{H_k\}$ is closed in the following sense. Any two spaces of the sequence generate a subsequence of $\{H_k\}$. We also derive representation theorems for bounded linear functionals over the spaces of the sequence $\{H_k\}$, which generalize the Lax representation theorem [3] and the Lax-Milgram theorem **[4]**; see also Landesman **[2]**.

2. **Generated spaces.** For fixed $x \in H_0$ we consider the linear functional over H_1, $g(y)=(y, x)_0$, $y \in H_1$. Then

$$|g(y)| = |(y, x)_0| \leqq \|x\|_0 \|y\|_0 \leqq \|x\|_0 \|y\|_1. \tag{2}$$

Hence $g(y)$ is a bounded linear functional over H_1. It follows from the Riesz representation theorem that there exists a unique $z \in H_1$ such that

$$g(y) = (y, x)_0 = (y, z)_1. \tag{3}$$

We define the negative norm of x by

$$\|x\|_{-1} = \|z\|_1. \tag{4}$$

To justify this designation of a norm we have to show that if $x_0 \in H_0$ with $\|x_0\|_{-1}=0$ then $x_0=0$. Indeed, $\|x_0\|_{-1}=0$ implies $(y, x_0)_0=0$ for all $y \in H_1$. Since H_1 is dense in H_0 it follows that $\|x_0\|_0=0$ and $x_0=0$.

H_{-1} is the space obtained by completion under the negative norm (4) of the elements of H_0. H_{-1} is isometrically isomorphic to a closed subspace, say $\tilde{H}_1$, of H_1. We show that $\tilde{H}_1$ exhausts all of H_1.

Lemma 1. $\tilde{H}_1=H_1$.

Proof. Let $y_0 \in H_1$ be orthogonal to $\tilde{H}_1$. Then $(y_0, z)_1=0$ for all $z \in \tilde{H}_1$. Therefore $(y_0, x)_0=0$ for all $x \in H_0$ and in particular for y_0. Hence $\|y_0\|_0=0$. Since H_1, as a vector space, is a subspace of H_0 we conclude that also $\|y_0\|_1=0$. This implies that $\tilde{H}_1=H_1$.

The inner product for elements w_1, w_2 in H_{-1} is defined by

$$(w_1, w_2)_{-1} = (y_1, y_2)_1 \tag{5}$$

where y_1, y_2 are the elements in H_1 corresponding respectively to w_1, w_2, in the preceding isometric isomorphism between H_1 and H_{-1}. This completes the construction of H_{-1} as a Hilbert space.

Definition. H_{-1} is defined as the *lower* space generated by H_0 and H_1.

Symbolically this will be denoted by

$$H_{-1} = H_0/H_1. \tag{6}$$

It follows from (3) that if $x \in H_0$, $y \in H_1$ then $|(y, x)_0| \leqq \|y\|_1 \|x\|_{-1}$. Hence the inner product in H_0 can be extended boundedly to $y \in H_1$ and $w \in H_{-1}$, such that $(y, w)_0$ satisfies the generalized Schwarz inequality

$$|(y, w)_0| \leqq \|y\|_1 \|w\|_{-1}. \tag{7}$$

Furthermore, if $w_1, w_2 \in H_{-1}$ correspond respectively to $y_1, y_2 \in H_1$, then

$$(w_1, w_2)_{-1} = (y_1, w_2)_0 = (y_1, y_2)_1. \tag{8}$$

From (8) follow the dual relationships for $w \in H_{-1}$ and $y \in H_1$,

$$\|w\|_{-1} = \sup_{Y \in H_1} |(Y, w)_0|/\|Y\|_1 \tag{9}$$

and

$$\|y\|_1 = \sup_{W \in H_{-1}} |(y, W)_0|/\|W\|_{-1}. \tag{10}$$

LEMMA 2. *If $x \in H_0$ then $\|x\|_{-1} \leqq \|x\|_0$.*

PROOF. If $z \in H_1$ corresponds to x as an element in H_{-1}, then $\|x\|_{-1}^2 = (x, x)_{-1} = (z, x)_0 \leqq \|z\|_0 \|x\|_0 \leqq \|z\|_1 \|x\|_0 = \|x\|_{-1} \|x\|_0$. Hence $\|x\|_{-1} \leqq \|x\|_0$.

Since H_0 is dense in H_{-1} it follows from Lemma 2 that H_{-1} and H_0 form a generating pair. They generate the lower space which we denote by H_{-2}, in the same way that H_{-1} is generated from H_0 and H_1. Therefore $H_{-2} = H_{-1}/H_0$. Successively we now construct the spaces $H_{-2}, H_{-3}, H_{-4}, \cdots$, such that H_{-k}, $k>0$, is the lower space generated by H_{-k+1} and H_{-k+2}; i.e. $H_{-k} = H_{-k+1}/H_{-k+2}$.

Next we construct from H_0 and H_1 the space H_2 as follows. H_2 is the subspace of H_1 for whose elements z there exist corresponding elements $x \in H_0$ such that, for all $y \in H_1$,

$$(z, y)_1 = (x, y)_0. \tag{11}$$

Since (11) can be satisfied for any $x \in H_0$, it follows that the correspondence between $z \in H_2$ and $x \in H_0$ is a mapping from H_2 onto H_0. Furthermore $x=0$ if and only if $z=0$ and therefore this is a one-to-one mapping.

The norm and inner product in H_2 are defined by

$$\|z\|_2 = \|x\|_0 \quad \text{and} \quad (z_1, z_2)_2 = (x_1, x_2)_0, \tag{12}$$

where $z, z_1, z_2 \in H_2$ correspond respectively to $x, x_1, x_2 \in H_0$.

It follows from (12) that the correspondence between H_0 and H_2 is an isometric isomorphism.

H_2 is dense in H_1. This follows from the fact that if $y \in H_1$ is orthogonal in H_1 to the elements of H_2, then (11) implies that $(x, y)_0 = 0$ for all $x \in H_0$. Hence $\|y\|_0 = 0$ and therefore $\|y\|_1 = 0$.

LEMMA 3. *If* $z \in H_2$ *then* $\|z\|_1 \leqq \|z\|_2$.

PROOF. If $x \in H_0$ corresponds to $z \in H_2$ then it follows from (11) that $\|z\|_1^2 = (z, z)_1 = (x, z)_0 \leqq \|x\|_0 \|z\|_0 \leqq \|x\|_0 \|z\|_1$. Hence $\|z\|_1 \leqq \|x\|_0 = \|z\|_2$.

DEFINITION. H_2 is defined as the *upper* space generated by H_0 and H_1. Symbolically this will be denoted by

$$H_2 = H_1/H_0. \tag{13}$$

Since H_2 is dense in H_1 it follows from Lemma 3 that H_1 and H_2 form a generating pair. We generate therefore successively the spaces H_2, H_3, $H_4, \cdots$. H_{k+2} is the upper space generated by H_k and H_{k+1}. Hence $H_{k+2} = H_{k+1}/H_k$, $k > 0$.

To complete the construction of all the generated spaces we will show that the processes of generating lower spaces and upper spaces are reciprocal to each other. We do this by showing that H_1 is the upper space generated by H_{-1} and H_0, and that H_0 is the lower space generated by H_1 and H_2.

LEMMA 4. $H_1 = H_0/H_{-1}$.

PROOF. Denote H_0/H_{-1} by H_1'. Then it is to be shown that $H_1' = H_1$. If $y \in H_1'$ corresponds to $w \in H_{-1}$ in the isometric isomorphism between H_{-1} and H_1', then $(w, x)_{-1} = (y, x)_0$ for all $x \in H_0$. Let W, $X \in H_1$ correspond respectively to w and x as elements in H_{-1}, in the isometric isomorphism between H_1 and H_{-1}. Then $(w, x)_{-1} = (W, X)_1 = (W, x)_0$. Hence $y = W$ and therefore $y \in H_1$. Also, y as an element both in H_1 and H_1' corresponds to the same element $w \in H_{-1}$. Conversely if $y \in H_1$ then for all $x \in H_0$, $(y, x)_0 = (y, X)_1 = (w, x)_{-1}$ and therefore $y \in H_1'$. It follows that $H_1 = H_1'$ which concludes the proof.

LEMMA 5. $H_0 = H_1/H_2$.

PROOF. Denote H_1/H_2 by H_0'. If $y \in H_1$, then for all $z \in H_2$, $(z, y)_1 = (z, Y)_2$ where $Y \in H_2$ corresponds to $y \in H_0'$ in the isometric isomorphism between H_2 and H_0'. Also, $(z, y)_1 = (x, y)_0 = (z, W)_2$ where x, $y \in H_0$ correspond respectively to z, $W \in H_2$ in the isometric isomorphism between H_2 and H_0. Hence $Y = W$. Therefore, y as an element in both H_0' and H_0 corresponds to the same element $Y \in H_2$. Since H_1 is dense in both H_0' and H_0, it follows that $H_0' = H_0$.

Summing up the properties of the spaces of the sequence $\{H_k\}$ we have:

THEOREM 1. *For any integer* k, $-\infty < k < \infty$,

(a) *The spaces* H_k *and* H_{k+1} *form a generating pair, with* H_k *the weaker and* H_{k+1} *the stronger space.*

(b) *$H_{k-1}=H_k/H_{k+1}$ and $H_{k+1}=H_k/H_{k-1}$.*

(c) *The inner product in H_k can be extended boundedly to elements $x\in H_{k-1}$ and $z\in H_{k+1}$, and $(z, x)_k$ satisfies the Schwarz inequality:*

$$|(z, x)_k| \leqq \|z\|_{k+1}\,\|x\|_{k-1}. \tag{14}$$

(d) *There exists an isometric isomorphism between H_{k-1} and H_{k+1}, uniquely determined by the relationships*

$$(x_1, x_2)_{k-1} = (z_1, x_2)_k = (z_1, z_2)_{k+1} \tag{15}$$

where $x_1, x_2\in H_{k-1}$ correspond to $z_1, z_2\in H_{k+1}$.

3. **Main theorems.** The correspondence of the elements $x\in H_{k-1}$ and $z\in H_{k+1}$ under the isometric isomorphism of Theorem 1 will be denoted by

$$(x; H_{k-1}) \leftrightarrow (z; H_{k+1}). \tag{16}$$

Now, (16) generates successively an isometric isomorphism between H_{k-m} and H_{k+m}, $m>0$, similarly denoted by $(z; H_{k-m})\leftrightarrow(Z; H_{k+m})$, through the mappings

$$(z; H_{k-m}) \leftrightarrow (z_1; H_{k-m+2}) \leftrightarrow \cdots \leftrightarrow (z_{m-1}; H_{k+m-2}) \leftrightarrow (Z; H_{k+m}). \tag{17}$$

LEMMA 6. *For integer k and positive integer m, the spaces H_k and H_{k+m} form a generating pair with H_k the weaker and H_{k+m} the stronger space.*

PROOF. H_{k+m}, as a linear vector space, is a subspace of H_k. Also, $\|z\|_k\leqq\|z\|_{k+m}$ for $z\in H_{k+m}$. Hence we only have to show that H_{k+m} is dense in H_k. Consider first H_{k+2}. Since H_{k+1} is dense in H_k and H_{k+2} is dense in H_{k+1}, it follows that for $\varepsilon>0$ and any $x\in H_k$ there exists $y\in H_{k+1}$ with $\|y-x\|_k<\varepsilon$, and $z\in H_{k+2}$ with $\|z-y\|_{k+1}<\varepsilon$. Hence $\|z-x\|_k\leqq\|z-y\|_k+\|y-x\|_k\leqq\|z-y\|_{k+1}+\|y-x\|_k<2\varepsilon$. Therefore H_{k+2} is dense in H_k. It follows similarly that H_{k+m} is dense in H_k which completes the proof.

Next we show that two corresponding elements in H_k and H_{k+2} are also corresponding elements in H_{k-1} and H_{k+1}.

LEMMA 7. *If $(x; H_k) \leftrightarrow (z; H_{k+2})$, then $(x; H_{k-1})\leftrightarrow(z; H_{k+1})$.*

PROOF. Let $Z\in H_{k+1}$ be such that $(x; H_{k-1})\leftrightarrow(Z; H_{k+1})$. We have to show that $Z=z$. For any $y\in H_{k+1}$ let $Y\in H_{k+2}$ correspond to y as an element in H_k, that is $(y; H_k)\leftrightarrow(Y; H_{k+2})$. It follows that $(x, y)_k=(Z, y)_{k+1}$ and also $(x, y)_k=(z, Y)_{k+2}=(z, y)_{k+1}$. Hence $(Z, y)_{k+1}=(z, y)_{k+1}$. Therefore $Z=z$.

From repeated application of Lemma 7 follows

LEMMA 8. *If $(x; H_k)\leftrightarrow(z; H_{k+2})$, then*

$$(x; H_{k-r}) \leftrightarrow (z; H_{k-r+2})\ \textit{for}\ r > 0.$$

LEMMA 9. *If* $x \in H_{k+r}$, $r>0$, *and* $(x; H_k) \leftrightarrow (z; H_{k+2})$, *then* $z \in H_{k+r+2}$ *and* $(x; H_{k+r}) \leftrightarrow (z; H_{k+r+2})$.

PROOF. Let $Z \in H_{k+r+2}$ be such that $(x; H_{k+r}) \leftrightarrow (Z; H_{k+r+2})$, then it follows from Lemma 8 that $(x; H_k) \leftrightarrow (Z; H_{k+2})$. Hence $Z=z$ which completes the proof.

Following Lemma 6, H_k and H_{k+m}, $m>0$, form a generating pair. We have therefore the lower space H_k/H_{k+m} and the upper space H_{k+m}/H_k. We now have the following result:

THEOREM 2. $H_k/H_{k+m} = H_{k-m}$.

PROOF. Denote $H_k/H_{k+m} = H'_{k-m}$. Then it is to be shown that $H'_{k-m} = H_{k-m}$. Let $x \in H_k$. Then there exists $y \in H_{k+m}$ such that

$$(x, w)_k = (y, w)_{k+m} \quad \text{for all } w \in H_{k+m}. \tag{18}$$

It follows from the definition of H'_{k-m} that $(x; H'_{k-m}) \leftrightarrow (y; H_{k+m})$. We shall show that also $(x; H_{k-m}) \leftrightarrow (y; H_{k+m})$. There exist

$$x_1 \in H_{k+1},\ x_2 \in H_{k+2},\ \cdots,\ x_{m-1} \in H_{k+m-1},\ x_m \in H_{k+m}$$

such that

$$(x, w)_k = (x_1, w)_{k+1} = (x_2, w)_{k+2} = \cdots = (x_{m-1}, w)_{k+m-1} = (x_m, w)_{k+m}$$

for all $w \in H_{k+m}$. It follows from (18) that $x_m = y$. Therefore

$$(y; H_{k+m}) \leftrightarrow (x_{m-1}; H_{k+m-2}) \quad \text{and} \quad (x_{m-1}; H_{k+m-1}) \leftrightarrow (x_{m-2}; H_{k+m-3}).$$

Hence from Lemma 7 $(x_{m-1}; H_{k+m-2}) \leftrightarrow (x_{m-2}; H_{k+m-4})$ and therefore $(y; H_{k+m}) \leftrightarrow (x_{m-2}; H_{k+m-4})$. Next, $(x_{m-2}; H_{k+m-2}) \leftrightarrow (x_{m-3}; H_{k+m-4})$ and hence $(x_{m-2}; H_{k+m-4}) \leftrightarrow (x_{m-3}; H_{k+m-6})$ and therefore $(y; H_{k+m}) \leftrightarrow (x_{m-3}; H_{k+m-6})$. Successively it follows that $(y; H_{k+m}) \leftrightarrow (x; H_{k-m})$.

Hence the elements of H_k as a subspace of H'_{k-m} are identical to these elements as a subspace of H_{k-m}. Since the elements of H_k are dense in H'_{k-m} by definition, and are dense in H_{k-m} by Lemma 6, it follows that $H'_{k-m} = H_{k-m}$.

COROLLARY 2.1. $H_{k+m}/H_k = H_{k+2m}$.

PROOF. Since Theorem 2 implies that $H_{k+m}/H_{k+2m} = H_k$, it follows from Lemma 4 that $H_{k+m}/H_k = H_{k+2m}$.

The following corollaries are an immediate consequence of Theorem 2.

COROLLARY 2.2. H_k *and* H_{k+m} *generate the sequence* $\{H_{k+sm}\}$, $-\infty < s < \infty$.

COROLLARY 2.3. *The inner product in* H_k *can be extended boundedly to* *elements* $x \in H_{k-m}$ *and* $z \in H_{k+m}$ *such that*

$$(x_1, x_2)_{k-m} = (z_1, x_2)_k = (z_1, z_2)_{k+m} \tag{19}$$

where $(x_1; H_{k-m}) \leftrightarrow (z_1; H_{k+m})$ *and* $(x_2; H_{k-m}) \leftrightarrow (z_2; H_{k+m})$.

We now derive two generalized representation theorems.

THEOREM 3 (LAX REPRESENTATION THEOREM). *For any integer k and positive integer m, all bounded linear functionals* $g(z)$ *over* H_{k+m} *are uniquely represented by*

$$g(z) = (z, x_g)_k, \qquad x_g \in H_{k-m}, \tag{20}$$

and all bounded linear functionals $h(x)$ *over* H_{k-m} *are uniquely represented by*

$$h(x) = \operatorname{conj}(z_h, x)_k, \qquad z_h \in H_{k+m}. \tag{21}$$

PROOF. By the Riesz representation theorem $g(z) = (z, z_g)_{k+m}$ with $z_g \in H_{k+m}$ uniquely determined. It follows from Corollary 2.3 that $g(z) = (z, x_g)_k$ where $(x_g; H_{k-m}) \leftrightarrow (z_g; H_{k+m})$, and x_g is uniquely determined in H_{k-m}. The representation $h(x) = \overline{(z_h, x)_k}$ is similarly established.

THEOREM 4 (GENERALIZED LAX-MILGRAM THEOREM). *Let* $B(z, x)$ *be a form defined for all elements* $z \in H_{k+m}$, $x \in H_{k-m}$, *which is linear in z, anti-linear in x, and satisfies* $|B(z, x)| \leqq M \|z\|_{k+m} \|x\|_{k-m}$ *for some constant M. Suppose that for some positive constant c, all pairs* z_0, x_0, *such that* $(z_0; H_{k+m}) \leftrightarrow (x_0; H_{k-m})$, *satisfy* $|B(z_0, x_0)| \geqq c\|z_0\|^2_{k+m} = c\|x_0\|^2_{k-m}$. *Then every bounded linear functional* $G(z)$ *over* H_{k+m} *admits a unique representation* $G(z) = B(z, x_G)$, $x_G \in H_{k-m}$, *and every bounded linear functional* $F(x)$ *over* H_{k-m} *admits a unique representation* $F(x) = \operatorname{conj} B(z_F, x)$, $z_F \in H_{k+m}$.

PROOF. We adapt the method of the proof of the Lax-Milgram theorem (see for example Nirenberg [5]) to the present circumstances. For fixed $x \in H_{k-m}$, $B(z, x)$ is a bounded linear functional over H_{k+m}. Hence by Theorem 3 there exists a unique $u \in H_{m-k}$ such that $B(z, x) = (z, u)_k$. This defines a linear mapping $u = \mathscr{A}x$ from H_{k-m} into itself. We substitute v for z, where $(v; H_{k+m}) \leftrightarrow (x; H_{k-m})$. This implies that $c\|x\|^2_{k-m} \leqq |B(v, x)| = |(v, u)_k| \leqq \|v\|_{k+m} \|u\|_{k-m} = \|x\|_{k-m} \|u\|_{k-m}$. Hence $\|x\|_{k-m} \leqq c^{-1} \|u\|_{k-m}$. It follows that the operator $\mathscr{A}$ has a bounded inverse. Therefore the range of $\mathscr{A}$ is closed and every element $x \in H_{k-m}$ corresponds to a unique u in the range of $\mathscr{A}$. To complete the proof it remains to show that the range of $\mathscr{A}$ is all of H_{k-m}. Assume that $v_0 \in H_{k+m}$ is orthogonal to the range of $\mathscr{A}$, that is $(v_0, u)_k = 0$, for all u in the range. Let $x_0 \in H_{k-m}$ be such that $(x_0; H_{k-m}) \leftrightarrow (v_0; H_{k+m})$ and let $u_0 = \mathscr{A}x_0$. Then $0 = |(v_0, u_0)_k| = |B(v_0, x_0)| \geqq c\|v_0\|^2_{k+m}$. Hence $\|v_0\|_{k+m} = 0$ and therefore the zero element is the only element in H_{k+m} orthogonal to the range of $\mathscr{A}$ (in H_{k-m}). It now follows from Corollary 2.3 that the range of $\mathscr{A}$ is all of H_{k-m}. The proof of the representation of the bounded linear functionals over H_{k-m} follows by a similar argument.

References

1. E. M. Landesman, *Hilbert-space methods in elliptic partial differential equations*, Pacific J. Math. **21** (1967), 113–131. MR **35** #806.

2. ———, *A generalized Lax-Milgram theorem*, Proc. Amer. Math. Soc. **19** (1968), 339–344. MR **37** #1965.

3. P. D. Lax, *On Cauchy's problem for hyperbolic equations and the differentiability of solutions of elliptic equations*, Comm. Pure Appl. Math. **8** (1955), 615–633. MR **17**, 1212.

4. P. D. Lax and A. N. Milgram, *Parabolic equations, contributions to the theory of partial differential equations*, Ann. of Math. Studies, no. 33, Princeton Univ. Press, Princeton, N.J., 1954, pp. 167–190. MR **16**, 709.

5. L. Nirenberg, *Remarks on strongly elliptic partial differential equations*, Comm. Pure Appl. Math. **8** (1955), 649–675. MR **17**, 742.

6. M. Schechter, *Negative norms and boundary problems*, Ann. of Math. (2) **72** (1960), 581–593. MR **23** #A2636.

7. K. Yosida, *Functional analysis*, Die Grundlehren der math. Wissenschaften, Band 123, Academic Press, New York; Springer-Verlag, Berlin, 1965. MR **31** #5054.

Department of Mathematics, Newark College of Engineering, Newark, New Jersey 07102