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HOMOMORPHISMS OF RINGS OF GERMS OF ANALYTIC FUNCTIONS

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ABSTRACT. Let S and S' be complex analytic manifolds with S Stein. Let $X\subset S$ and $X'\subset S'$ be compact sets with X holomorphically convex. Denote by $\mathcal{O}(X)$ (respectively $\mathcal{O}(X')$) the ring of germs on X (respectively X') of functions analytic near X (respectively X'). It is shown that each nonzero homomorphism of $\mathcal{O}(X)$ into $\mathcal{O}(X')$ is given by composition with an analytic map defined in a neighborhood of X' and taking values in S.

If S and S' are complex analytic manifolds, then every analytic mapping of S into S' induces (via composition) a homomorphism of the ring of analytic functions on S' into the ring of analytic functions on S. It is an important and deep result that the converse is also true, providing that S is a Stein manifold (see [**1**]). In this note we obtain an analogous result for homomorphisms of rings of germs of analytic functions on compact subsets of a complex analytic manifold. We show that each such homomorphism is given by composition with an analytic mapping.

1. **Preliminaries and notation.** Let S be a Stein manifold and U an open subset of S. We denote by $\mathcal{O}(U)$ the ring of analytic functions on U. It is well known (see [**1**], for example) that each nonzero complex-valued homomorphism of $\mathcal{O}(U)$ is continuous with respect to the topology on $\mathcal{O}(U)$ of uniform convergence on compact subsets of U. We denote the space of such homomorphisms by $\Delta\mathcal{O}(U)$ If $f\in\mathcal{O}(U)$ then $f^\wedge$ denotes the function on $\Delta\mathcal{O}(U)$ defined by $f^\wedge(\alpha)=\alpha(f)$ for each α in $\Delta\mathcal{O}(U)$.

Since S is Stein, $\Delta\mathcal{O}(S)=S$, so we have the natural restriction map $\pi_U:\Delta\mathcal{O}(U)\to S$ given by $\pi_U(\alpha)(f)=\alpha(f|U)$. Rossi [**4**] has shown that $\Delta\mathcal{O}(U)$ admits the structure of a Stein manifold in such a way that: (i) the evaluation map $U\to\Delta\mathcal{O}(U)$ is a biholomorphism of U with an open subset of $\Delta\mathcal{O}(U)$ (we will regard U as an open subset of $\Delta\mathcal{O}(U)$); (ii) if $f\in\mathcal{O}(U)$ then $f^\wedge$ is the unique analytic extension of f to $\Delta\mathcal{O}(U)$ (so that $\mathcal{O}(\Delta\mathcal{O}(U))=\mathcal{O}(U)$); (iii) π_U is locally a biholomorphism.

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If X is a compact subset of S, we denote by $\mathcal{O}(X)$ the ring of germs on X of functions analytic near X. If f is analytic in a neighborhood of X, we denote its germ on X by f; f is called a *representative* of f. We will also regard a germ in $\mathcal{O}(X)$ as a continuous function on X. It was shown in [2] and [6] that each nonzero complex-valued homomorphism on $\mathcal{O}(X)$ is continuous, relative to the natural inductive limit topology on $\mathcal{O}(X)$. We say that X is *holomorphically convex* if each such homomorphism is given by evaluation at a point of X. Equivalently, X is holomorphically convex if and only if $\{\pi_U(\Delta\mathcal{O}(U)); U$ is open and $X\subset U\}$ is a fundamental system of neighborhoods of X. More information about holomorphically convex sets may be found in [2] and [5]. We refer to [1] for general information about several complex variables.

2. **Main result.** Now let S and S' be complex analytic manifolds with S Stein, and let $X\subset S$ and $X'\subset S'$ be compact sets with X holomorphically convex. We wish to study homomorphisms of $\mathcal{O}(X)$ into $\mathcal{O}(X')$. If F is an analytic map of a neighborhood U' of X' into S such that $F(X')\subset X$, then F induces a homomorphism $\Phi_F:\mathcal{O}(X)\to\mathcal{O}(X')$ as follows. Let f be in $\mathcal{O}(X)$. Choose an open set U containing X and a representative f of f which is analytic on U. Then $f\circ F$ is analytic on a neighborhood of X'; we let $\Phi_F(f)$ be the germ of $f\circ F$ on X'. By a straightforward calculation, we may verify that Φ_F is a well-defined homomorphism of $\mathcal{O}(X)$ into $\mathcal{O}(X')$. Our main result is that every homomorphism arises in this way.

THEOREM. *Let $\Phi:\mathcal{O}(X)\to\mathcal{O}(X')$ be a nonzero homomorphism. Then there is an open set U' containing X' and an analytic function $F: U'\to S$ such that $F(X')\subset X$ and $\Phi=\Phi_F$. The germ of F on X' is unique.*

For the proof of this theorem we shall make use of two lemmas. If T is a subset of S and $\mathcal{F}$ is a subset of $\mathcal{O}(S)$, we say that $\mathcal{F}$ separates points on T if for each x and y in T with $x\neq y$ there is a function $f\in\mathcal{F}$ such that $f(x)\neq f(y)$. If the (complex) dimension of S is n, we say that $\mathcal{F}$ provides local coordinates on T if for each x in T there are n functions $f_1, \cdots, f_n$ in $\mathcal{F}$ such that $df_1\wedge\cdots\wedge df_n(x)\neq 0$.

LEMMA 1. *Let $\{f_1, \cdots, f_k\}$ be a subset of $\mathcal{O}(S)$ which separates points and provides local coordinates on X, and let U be an open set containing X. Then there is an open set W containing X such that for each x in X, each integer M and each f in $\mathcal{O}(U)$ vanishing to total order at least M at x, there are functions $g_1, \cdots, g_N$ in $\mathcal{O}(W)$ and monomials $h_1, \cdots, h_N$ of degree M in $f_1-f_1(x), \cdots, f_k-f_k(x)$ such that $f=\sum g_i h_i$ in W.*

PROOF. A straightforward compactness argument shows that there is an open set U_1 containing X such that $\{f_1, \cdots, f_k\}$ separates points and provides local coordinates on U_1. Since X is holomorphically convex, there is an open set W containing X such that $\pi_W(\Delta\mathcal{O}(W)) \subset U \cap U_1$. Let x be a point of X, M an integer, and $h_1, \cdots, h_N$ the monomials of degree M in $f_1 - f_1(x), \cdots, f_k - f_k(x)$.

Let $\mathcal{O}$ be the sheaf of germs of analytic functions on $\Delta\mathcal{O}(W)$ and $\mathcal{O}^N$ the N-fold Cartesian product. Let $\mathscr{I}$ be the ideal sheaf of the discrete variety $\pi_W^{-1}(x)$ and $\mathscr{I}_M$ the sheaf of ideals generated by M-fold products of elements of $\mathscr{I}$. For each i, let $\phi_i = (h_i|W)^\wedge$ and let $\mu : \mathcal{O}^N \to \mathscr{I}_M$ be given by $\mu(\gamma_1, \cdots, \gamma_N) = \sum \phi_i \gamma_i$. Observe that the functions

$$((f_1 - f_1(x)) \mid W)^\wedge, \cdots, ((f_k - f_k(x)) \mid W)^\wedge$$

provide local coordinates on $\Delta\mathcal{O}(W)$ and vanish simultaneously only on $\pi_W^{-1}(x)$. Thus at each point of $\pi_W^{-1}(x)$, each germ in $\mathscr{I}_M$ can be expressed locally as a power series in these functions, while at each point not in $\pi_W^{-1}(x)$ at least one of these functions is locally a unit. It follows that μ is surjective.

This gives rise to the following short exact sequence of sheaves over $\Delta\mathcal{O}(W)$:

$$0 \to \ker \mu \to \mathcal{O}^N \to \mathscr{I}_M \to 0.$$

This induces a long exact cohomology sequence, the relevant terms of which are:

$$H^0(\Delta\mathcal{O}(W), \mathcal{O}^N) \xrightarrow{\mu^*} H^0(\Delta\mathcal{O}(W), \mathscr{I}_M) \to H^1(\Delta\mathcal{O}(W), \ker \mu).$$

Since $\Delta\mathcal{O}(W)$ is a Stein manifold and $\ker \mu$ is a sheaf of relations on $\Delta\mathcal{O}(W)$, it follows from Cartan's Theorem B that $H^1(\Delta\mathcal{O}(W), \ker \mu) = 0$, so that μ^* is surjective.

Finally, if $f \in \mathcal{O}(U)$ vanishes to total order at least M at x, then $(f|W)^\wedge = f \circ \pi_W$ vanishes to total order at least M at each point of $\pi_W^{-1}(x)$. Thus $(f|W)^\wedge \in H^0(\Delta\mathcal{O}(W), \mathscr{I}_M)$. Thus $(f|W)^\wedge$ is the image under μ^* of an N-tuple of functions in $\mathcal{O}(\Delta\mathcal{O}(W)) = \mathcal{O}(W)$, which is the desired result.

LEMMA 2. *Let $\{f_1, \cdots, f_k\}$ be a subset of $\mathcal{O}(S)$ which separates points and provides local coordinates on X. If Φ_1 and Φ_2 are nonzero homomorphisms of $\mathcal{O}(X)$ into $\mathcal{O}(X')$ such that $\Phi_1(f_i) = \Phi_2(f_i)$ for each i, then $\Phi_1 = \Phi_2$.*

PROOF. Since every nontrivial homomorphism of $\mathcal{O}(X)$ into $\mathbf{C}$ is given by evaluation at a point of X, we can define maps $\Phi_j^* : X' \to X$ by requiring that for each $f \in \mathcal{O}(X)$ and $y \in X'$, $\Phi_j^*(y)(f) = \Phi_j(f)(y)$ for $j = 1, 2$. For each i and each $y \in X'$ we have:

$$f_i(\Phi_1^*(y)) = \Phi_1(f_i)(y) = \Phi_2(f_i)(y) = f_i(\Phi_2^*(y)).$$

Since $\{f_1, \cdots, f_k\}$ separates points on X, it follows that $\Phi_1^* = \Phi_2^*$.

Let f be in $\mathcal{O}(X)$. In order to show that $\Phi_1(f)=\Phi_2(f)$ it suffices to show that $\Phi_1(f)-\Phi_2(f)$ vanishes to arbitrarily high order at each point of X'. To this end, let M be an integer and y a point of X'; set $x=\Phi_1^*(y)=\Phi_2^*(y)$. In a neighborhood of x, some representative of f may be represented by a power series in $f_1-f_1(x), \cdots, f_k-f_k(x)$; let P_M denote the sum of the terms of this series whose total order in $f_1-f_1(x), \cdots, f_k-f_k(x)$ does not exceed $M-1$. Hence the germ $\boldsymbol{f}-\boldsymbol{P}_M$ vanishes to total order at least M at x. In view of Lemma 1, we may find germs $\boldsymbol{g}_1, \cdots, \boldsymbol{g}_N$ in $\mathcal{O}(X)$ and monomials $\boldsymbol{h}_1, \cdots, \boldsymbol{h}_N$ of order M in $f_1-f_1(x), \cdots, f_k-f_k(x)$ such that $\boldsymbol{f}-\boldsymbol{P}_M=\sum \boldsymbol{g}_i\boldsymbol{h}_i$.

Since Φ_1 and Φ_2 are nonzero, it follows that $\Phi_1(\mathbf{1})=\Phi_2(\mathbf{1})=\mathbf{1}$, so that $\Phi_1(\boldsymbol{P}_M)=\Phi_2(\boldsymbol{P}_M)$. Hence

$$\begin{aligned}\Phi_1(\boldsymbol{f}) - \Phi_2(\boldsymbol{f}) &= \Phi_1(\boldsymbol{f} - \boldsymbol{P}_M) - \Phi_2(\boldsymbol{f} - \boldsymbol{P}_M)\\ &= \sum \Phi_1(\boldsymbol{h}_j)\{\Phi_1(\boldsymbol{g}_j) - \Phi_2(\boldsymbol{g}_j)\}\end{aligned}$$

since $\Phi_1(\boldsymbol{h}_j)=\Phi_2(\boldsymbol{h}_j)$. Since $\boldsymbol{h}_j$ is the product of M germs that vanish at x, $\Phi_1(\boldsymbol{h}_j)$ is the product of M germs that vanish at y. It follows that $\Phi_1(\boldsymbol{f})-\Phi_2(\boldsymbol{f})$ vanishes to order at least M at y. This completes the proof.

Proof of the Theorem. By the imbedding theorem for Stein manifolds, we can find functions $g_1, \cdots, g_k$ in $\mathcal{O}(S)$ such that $G=(g_1, \cdots, g_k)$ is a regular proper imbedding of S as a closed submanifold of C^k. We can find an open set Ω containing $G(S)$ and a holomorphic retraction $h:\Omega\to G(S)$. If f is analytic in an open set containing $G(X)$, then $f\circ G$ is analytic in an open set containing X, so we have defined a homomorphism $\nu:\mathcal{O}(G(X))\to\mathcal{O}(X)$. Observe that $\nu(\boldsymbol{f})=\nu(\boldsymbol{g})$ whenever $\boldsymbol{f}$ and $\boldsymbol{g}$ have representatives that agree in a $G(S)$-neighborhood of $G(X)$. Let $\Phi'=\Phi\circ\nu$ and set $\boldsymbol{\varphi}_j=\Phi'(\boldsymbol{z}_j)=\Phi(\boldsymbol{g}_j)$.

We can find an open set U' containing X' and functions $\varphi_1, \cdots, \varphi_k$ in $\mathcal{O}(U')$ which represent $\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_k$. Set $\varphi=(\varphi_1, \cdots, \varphi_k): U'\to C^k$ and consider the homomorphism $\Phi_\varphi:\mathcal{O}(G(X))\to\mathcal{O}(X')$. It is easy to see that $\Phi_\varphi(\boldsymbol{z}_i)=\boldsymbol{\varphi}_i$ for each i. Since X is holomorphically convex in S, G is an imbedding and $G(S)$ is closed in C^k, it follows that $G(X)$ is homomorphically convex in C^k (see [5]). From Lemma 2 we see that $\Phi_\varphi=\Phi'$.

Now set $F=G^{-1}\circ h\circ\varphi: U'\to S$. If $\boldsymbol{f}_i$ denotes the germ of $g_i\circ G^{-1}\circ h$ on $G(X)$, then we see that $\Phi'(\boldsymbol{f}_i)=\Phi'(\boldsymbol{z}_i)=\Phi(\boldsymbol{g}_i)$. Hence $\Phi_F(\boldsymbol{g}_i)=\Phi'(\boldsymbol{f}_i)=\Phi(\boldsymbol{g}_i)$. Using Lemma 2 again, it follows that $\Phi=\Phi_F$. (Observe that this argument shows that $\varphi(U')\subset G(S)$ so that $F=G^{-1}\circ h\circ\varphi=G^{-1}\circ\varphi$.) It is easy to see that $F|X'=\Phi^*$ so that $F(X')\subset X$. Finally, a straightforward calculation shows that Φ_F depends only on the germ of F on X' and that functions representing different germs induce distinct homomorphisms.

3. **Remarks.** It is tempting to try to generalize the theorem to the case in which X is not holomorphically convex by passing to the space of non-zero homomorphisms of $\mathcal{O}(X)$ into C. In general, however, this space admits no natural imbedding into a Stein manifold (see [**2**] and [**6**]).

If S' is Stein and X' is holomorphically convex, then $\Phi = \Phi_F$ is an isomorphism if and only if F is a biholomorphism of a neighborhood of X' with a neighborhood of X.

The author does not know whether the theorem remains true in the context of analytic spaces, rather than manifolds.

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