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COHOMOLOGY OPERATIONS ON p-FOLD SUMS

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ABSTRACT. A formula is given for evaluating higher order cohomology operations on integral classes that are p-fold multiples, p a prime.

Let Ω be an nth order ($n \geqq 2$) cohomology operation defined on integral cohomology classes, given by a relation

$$\sum_i \alpha_i \Phi_i = 0, \tag{1}$$

where each Φ_i is an $(n-1)$st order (integral) operation and each α_i is an element of the mod p Steenrod algebra $\mathscr{A}$. We will think of each operation Φ_i as being defined on integral classes of some fixed degree, q. We suppose that Φ_i is the suspension of an operation Ψ_i, defined on classes of degree $q+1$, and that relation (1) desuspends. In particular, each Φ_i is additive. Suppose now that X is a space and $u \in H^q(X, Z)$ is a class such that $\Phi_i(u)$ is defined for each i. Of course Ω is not necessarily defined on u, but Ω is defined on pu, since each Φ_i is additive. Our problem is: how does one compute $\Omega(pu)$?

In [2] a morphism $\varepsilon: \mathscr{A} \to \mathscr{A}$ is defined, of degree -1, characterized by the following properties:

(i) ε is a derivation of the graded algebra $\mathscr{A}$.

(ii) If $p=2$, $\varepsilon(Sq^n)=Sq^{n-1}$, $n \geqq 1$; if $p>2$, $\varepsilon(\beta_p)=1$, $\varepsilon(P^i)=0$, $i \geqq 0$, where β_p denotes the mod p Bockstein. (For $p=2$, ε is the morphism $\hat{\kappa}$ considered by Kristensen [1].)

We now can state our result.

THEOREM. *Let Ω be an operation associated with relation* (1) *and let $u \in H^q(X; Z)$ be a class in the domain of each Φ_i. Then,*

$$(-1)^s \sum \varepsilon(\alpha_i)\Phi_i(u) \subset \Omega(pu),$$

where $s=q+\deg \Omega$.

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PROOF. Let B denote a universal example for the operations $\{\Psi_i\}$ defined on integral classes of degree $q+1$. Thus, B is an $(n-1)$st stage Postnikov system over $K(Z, q+1)$ (note [3]). By hypothesis, there are classes $\psi_i \in H^*(B; Z_p)$ such that

$$\sum (-1)^{a_i} \alpha_i \psi_i = 0, \quad \text{where } a_i = \deg \alpha_i.$$

Suppose $d_i = \deg \psi_i$, and set $C = \times_i K(Z_p, q+1+d_i)$. If we let $\psi = \{\psi_i\}$, we then have a map $\psi: B \to C$. Denote by

$$\Omega C \xrightarrow{i} E \xrightarrow{\pi} B \tag{2}$$

the principal fibration with ψ as classifying map. By hypothesis, there is a class $\lambda \in H^{s+1}(E; Z_p)$, thought of as a map

$$E \xrightarrow{\lambda} K(Z_p, s+1) = K,$$

such that $i^*\lambda = \alpha = \{\alpha_i\} \in H^{s+1}(\Omega C; Z_p)$.

Now take the loops of (2); we obtain a fibration

$$\Omega^2 C \xrightarrow{i'} \Omega E \xrightarrow{\pi'} \Omega B,$$

and if we denote by σ the loop homomorphism in cohomology, then $\sigma\psi = \phi = (\phi_i)$, and $\sigma\lambda = \omega$, a representative for Ω.

Consider the following commutative diagram:

$$\begin{tikzcd}
\Omega E \arrow[r, "\pi'"] \arrow[d, "\omega"] & \Omega B \arrow[r, "\phi"] \arrow[d, "t"] & \Omega C \arrow[r, "i"] \arrow[d, equal] & E \arrow[d, "\lambda"] \\
\Omega K \arrow[r, "k"] & \Omega L \arrow[r, "j"] & \Omega C \arrow[r, "\alpha"] & K
\end{tikzcd} \tag{3}$$

Here the lower line is the principal fibration sequence for the map α. (ΩL is a loop space since α is stable.) Since the right-hand square commutes, the map t exists. Now apply the functor $[X, \;]$ to (3); we obtain the following commutative diagram with each row an exact sequence:

$$\begin{tikzcd}
& {[X, \Omega E]} \arrow[r, "\pi'_*"] \arrow[d, "\omega_*"] & {[X, \Omega B]} \arrow[r, "\phi_*"] \arrow[d, "t_*"] & {[X, \Omega C]} \arrow[d, equal] & \\
& H^s(X; Z_p) \arrow[r, "k_*"] & {[X, \Omega L]} \arrow[r, "j_*"] & {[X, \Omega C]} \arrow[r, "\alpha_*"] & H^{s+1}(X; Z_p)
\end{tikzcd}$$

By hypothesis, there is a class $v \in [X, \Omega B]$ such that v goes to u in $H^q(X; Z)$. Since $p[X, \Omega C] = 0$, by exactness there is a class $x \in [X, \Omega E]$ such that $\pi'_*(x) = pv$. And by definition $\omega_*(x) \in \Omega(pu)$.

On the other hand, consider $y = t_*(v) \in [X, \Omega L]$. We have $j_*(y) = \phi_*(v) = \{\phi_{i*}(v)\}$ where $\phi_{i*}(v) \in \Phi_i(u)$. Also, $k_* \omega_*(x) = py$. Thus, by Corollary 3.7 of [**2**],

$$\omega_*(x) = (-1)^s \sum_i \varepsilon(\alpha_i) \phi_{i*}(v),$$

which completes the proof.

EXAMPLE. The simplest example of interest is the relation

$$Sq^2 Sq^2 = 0, \quad \text{on integral classes.} \tag{4}$$

Since $\varepsilon(Sq^2) = Sq^1$, we then have the result:

$$Sq^3 H^*(X; Z) \subset \Omega(2H^*(X; Z)),$$

where Ω is the secondary operation given by (4).

REMARK. As pointed out by F. Peterson, if (1) is in fact a relation that holds on mod p classes, then $\Omega(pu) \equiv 0$. For let ρ denote the cohomology homomorphism induced by the coefficient group epimorphism $Z \to Z_p$. Then for any integral class v in the domain of Ω, $\Omega(v) \equiv \Omega(\rho v)$, and hence

$$\Omega(pu) \equiv \Omega(\rho(pu)) \equiv \Omega(0) \equiv 0.$$

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