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NOTE ON THE PROJECTIVE LIMIT ON SMALL CATEGORIES

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ABSTRACT. Let X be a small category and let $\mathbf{Ab}^X$ be the category of covariant functors on X with values in $\mathbf{Ab}$. Consider the projective limit functor $\operatorname{proj\,lim}_X : \mathbf{Ab}^X \to \mathbf{Ab}$. The categories X for which $\operatorname{proj\,lim}_X$ is exact are characterized, proving a conjecture of Oberst.

In [Bull. Amer. Math. Soc. **74** (1968), 1129–1132], U. Oberst formulated a conjecture on the exactness of the projective limit functor on the category of functors on a small category with values in the category of abelian groups.

In this note we give a proof of his conjecture. Some of the lemmas seem to have been proved by U. Oberst and J. R. Isbell by other methods.

THEOREM. *If X is a small connected category and* $\mathbf{Ab}$ *is the category of abelian groups, then the two following conditions are equivalent*:

(i) *For all* $F \in \operatorname{ob} \mathbf{Ab}^X$, $\operatorname{proj\,lim}^{(i)} F = 0$, *for all* $i \geq 1$.

(ii) *There exists* $y \in \operatorname{ob} X$ *such that*

(1) *for all x there exists* $\xi \in X(y, x)$;

(2) *every diagram*

$$\begin{tikzcd} x \arrow[dr] & & x' \arrow[dl] \\ & u & \end{tikzcd}$$

in X can be completed to a commutative diagram

$$\begin{tikzcd} & y \arrow[dl] \arrow[dr] & \\ x \arrow[dr] & & x' \arrow[dl] \\ & u & \end{tikzcd}$$

in X;

(3) *there exists* $\varepsilon \in X(y, y)$ *such that, for all* $\xi \in X(y, y)$, $\xi\varepsilon = \varepsilon$.

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PROOF. Since (ii) implies that

$$\operatorname{proj\,lim} F = H^0(X(y, y), F(y)) = \{\alpha \in F(y) \mid F(\varepsilon)(\alpha) = \alpha\}$$
$$= \{F(\varepsilon)(\beta) \mid \beta \in F(y)\},$$

it is trivial to see that (ii)$\Rightarrow$(i).

To prove that (i) implies (ii), let F be the object of $\mathbf{Ab}^X$ defined by $F(x)=\coprod_{\xi\in\mathrm{ob}(X/x)} \boldsymbol{Z}\xi$ with $\boldsymbol{Z}\xi=\boldsymbol{Z}$ for all $\xi \in \mathrm{ob}(X/x)$.

Consider the obvious epimorphism $\rho: F\to \boldsymbol{Z}$ with $\boldsymbol{Z}$ the constant object of $\mathbf{Ab}^X$. Since proj lim is exact we have that $\rho^*:\operatorname{proj\,lim} F\to\operatorname{proj\,lim} \boldsymbol{Z}=\boldsymbol{Z}$ is an epimorphism. Therefore there exists $\alpha \in \operatorname{proj\,lim} F$ with $\rho^*(\alpha)=1$.

If $\pi_x:\operatorname{proj\,lim} F\to F(x)$ is the canonical homomorphism, then for all $x \in X$, $\alpha_x=\pi_x(\alpha) \in F(x)$ is nonzero. Now

$$\alpha_x = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_x^j(y_i)\xi_{ij}^x, \quad \text{where } \xi_{ij}^x \in X(y_i, x),\ \alpha_x^j(y_i) \in \boldsymbol{Z}, \text{ and } \sum_{i,j} \alpha_x^j(y_i) = 1.$$

For at least one i we must have $\sum_{j=1}^m \alpha_x^j(y_i)\neq 0$ and we may assume that $\alpha_x^j(y_i)\neq 0$ for $1\leqq j\leqq m'\leqq m$.

If the diagram

$$\begin{tikzcd} x \arrow[dr, "\varphi"'] & & x' \arrow[dl, "\psi"] \\ & u & \end{tikzcd}$$

is in X, then $\varphi^*\alpha_x=\alpha_u=\psi^*\alpha_{x'}$ and therefore

$$\sum_j \alpha_x^j = \sum_j \alpha_u^j = \sum \alpha_{x'}^j, \quad \text{with } \alpha_*^j = \alpha_*^j(y_i).$$

Since X is connected it follows that $\alpha_x^1(y_i)\neq 0$ for all $x \in X$ and there exist $\xi_{ij}^u \in X(y_i, u)$ with the corresponding $\alpha_u^j(y_i)\neq 0$. Consequently there exist $\xi_{ij}^x \in X(y_i, x)$, $\xi_{ij}^{x'} \in X(y_i, x')$ with $\varphi\xi_{ij}^x=\xi_{ij}^u=\psi\xi_{ij}^{x'}$, i.e. the above diagram can be completed to

$$\begin{tikzcd} & y_i \arrow[dl, "\xi_{ij}^x"'] \arrow[dr, "\xi_{ij}^{x'}"] & \\ x \arrow[dr, "\varphi"'] & & x' \arrow[dl, "\psi"] \\ & u & \end{tikzcd}$$

We have proved (ii)(2) and at the same time (ii)(1). We need only prove (ii)(3). Let F_1 be the object of $\mathbf{Ab}^X$ defined by

$$F_1(x) = \coprod_{\xi \in X(y,x)} Z\xi$$

with $y=y_i$ (i.e. the y_i picked above).

By (ii)(1) there exists an epimorphism $F_1 \to \boldsymbol{Z}$ in $\mathbf{Ab}^X$. Since, by assumption, $\boldsymbol{Z}$ is projective as an object of $\mathbf{Ab}^X$, $\boldsymbol{Z}$ is a direct summand of F_1, therefore $\boldsymbol{Z}$ is a direct summand of $F_1(y)$, as an $X(y, y)$-module. But $F_1(y)$ can be identified with the monoid algebra $\boldsymbol{Z}[X(y, y)]$ and it therefore follows that the cohomology of the monoid $M=X(y, y)$ is trivial.

LEMMA A. *If a monoid M is cohomologically trivial, then there exists an $\varepsilon \in M$ such that, for all $\xi \in M$, $\xi\varepsilon=\varepsilon$.*

PROOF. Consider the epimorphism $\boldsymbol{Z}[M] \to \boldsymbol{Z}$. Since cohomology is trivial, the corresponding homomorphism $H^0(M, \boldsymbol{Z}(M)) \to H^0(M, \boldsymbol{Z})=\boldsymbol{Z}$ is an epimorphism. Now $H^0(M, \boldsymbol{Z}(M))=\{\sum_{i=1}^m \alpha_i\xi_i | \alpha_i \in \boldsymbol{Z},\ \xi_i \in M$ such that, for all $\xi \in M$ and all i, $\xi\xi_i=\xi_j$ for some $j=j_\xi$ with $\alpha_i=\alpha_{j_\xi}\}$.

It follows that there exists an element $\sum_{i=1}^n \alpha_i\xi_i \in \boldsymbol{Z}(M)$ with $\sum_{i=1}^n \alpha_i=1$ such that for all $\xi \in M$, $\xi\xi_i=\xi_{\sigma_\xi(i)}$ where σ_ξ is a permutation of $\{1, 2, \cdots, n\}$.

Let $S(n)$ be the symmetric group, then the correspondence $\xi \to \sigma_\xi$ gives a homomorphism $\sigma: M \to S(n)$ since $\xi'\xi\xi_i=\xi'\xi_{\sigma_\xi(i)}=\xi_{\sigma_{\xi'}(\sigma_\xi(i))}$.

Let $H=\operatorname{im}\sigma$ then H is a subgroup of $S(n)$. This follows from the fact that by the theorem of Lagrange, any submonoid of a finite group is a subgroup.

SUBLEMMA B. *If $M \xrightarrow{\sigma} H$ is a surjective homomorphism of monoids and if M is cohomologically trivial, then H is cohomologically trivial.*

PROOF. This follows from $H^0(M, -)=H^0(H, -)$ in the category of H-modules. Q.E.D.

SUBLEMMA C. *If a group G is cohomologically trivial, then $G=\{1\}$.*

PROOF. As above there exists an element $\sum_{i=1}^n \alpha_i\xi_i \in \boldsymbol{Z}(G)$ such that for all $\xi \in G$, $\xi\xi_i=\xi_{\sigma_\xi(i)}$ and $\alpha_i=\alpha_{\sigma_\xi(i)}$, $\sum_{i=1}^n \alpha_i=1$. Since for all i, j there exists an element ξ with $\sigma_\xi(i)=j$, we have $\alpha_i=\alpha_j$ for all i, j and $G=\{\xi_1, \cdots, \xi_n\}$. Since

$$1 = n\cdot\alpha_1 = |G|\cdot\alpha_1$$

it follows that $|G|=1$ and $\alpha_1=1$, and consequently $G=\{1\}$. Q.E.D.

Combining B and C we find that $\sigma_\xi=1$ for all $\xi \in M$. This of course means that for all $\xi \in M$, $\xi\xi_i=\xi_i$. Put $\varepsilon=\xi_i$ for some i, and we have proved A. Q.E.D.

This ends the proof of the Theorem since A implies (ii)(3). Q.E.D.

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