

NOTE ON THE PROJECTIVE LIMIT ON SMALL CATEGORIES

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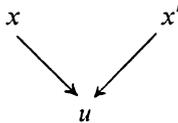
ABSTRACT. Let X be a small category and let \mathbf{Ab}^X be the category of covariant functors on X with values in \mathbf{Ab} . Consider the projective limit functor $\text{proj lim}_X: \mathbf{Ab}^X \rightarrow \mathbf{Ab}$. The categories X for which proj lim_X is exact are characterized, proving a conjecture of Oberst.

In [Bull. Amer. Math. Soc. **74** (1968), 1129–1132], U. Oberst formulated a conjecture on the exactness of the projective limit functor on the category of functors on a small category with values in the category of abelian groups.

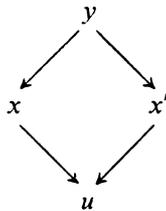
In this note we give a proof of his conjecture. Some of the lemmas seem to have been proved by U. Oberst and J. R. Isbell by other methods.

THEOREM. *If X is a small connected category and \mathbf{Ab} is the category of abelian groups, then the two following conditions are equivalent:*

- (i) *For all $F \in \text{ob } \mathbf{Ab}^X$, $\text{proj lim}^{(i)} F = 0$, for all $i \geq 1$.*
- (ii) *There exists $y \in \text{ob } X$ such that*
 - (1) *for all x there exists $\xi \in X(y, x)$;*
 - (2) *every diagram*



in X can be completed to a commutative diagram



in X ;

- (3) *there exists $\varepsilon \in X(y, y)$ such that, for all $\xi \in X(y, y)$, $\xi\varepsilon = \varepsilon$.*

Received by the editors August 2, 1971.

AMS 1970 subject classifications. Primary 18A30, 18A25; Secondary 18E25.

Key words and phrases. Category of functors, projective limit, cohomologically trivial monoid, cohomologically trivial group.

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PROOF. Since (ii) implies that

$$\begin{aligned} \text{proj lim } F &= H^0(X(y, y), F(y)) = \{\alpha \in F(y) \mid F(\varepsilon)(x) = \alpha\} \\ &= \{F(\varepsilon)(\beta) \mid \beta \in F(y)\}, \end{aligned}$$

it is trivial to see that (ii) \Rightarrow (i).

To prove that (i) implies (ii), let F be the object of \mathbf{Ab}^X defined by $F(x) = \coprod_{\xi \in \text{ob}(X/x)} Z\xi$ with $Z\xi = Z$ for all $\xi \in \text{ob}(X/x)$.

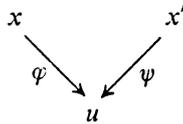
Consider the obvious epimorphism $\rho: F \rightarrow Z$ with Z the constant object of \mathbf{Ab}^X . Since proj lim is exact we have that $\rho^*: \text{proj lim } F \rightarrow \text{proj lim } Z = Z$ is an epimorphism. Therefore there exists $\alpha \in \text{proj lim } F$ with $\rho^*(\alpha) = 1$.

If $\pi_x: \text{proj lim } F \rightarrow F(x)$ is the canonical homomorphism, then for all $x \in X$, $\alpha_x = \pi_x(\alpha) \in F(x)$ is nonzero. Now

$$\alpha_x = \sum_{i=1}^n \sum_{j=1}^m \alpha_x^j(y_i) \xi_{ij}^x, \quad \text{where } \xi_{ij}^x \in X(y_i, x), \alpha_x^j(y_i) \in Z, \text{ and } \sum_{i,j} \alpha_x^j(y_i) = 1.$$

For at least one i we must have $\sum_{j=1}^m \alpha_x^j(y_i) \neq 0$ and we may assume that $\alpha_x^j(y_i) \neq 0$ for $1 \leq j \leq m' \leq m$.

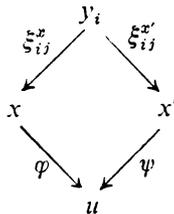
If the diagram



is in X , then $\varphi^* \alpha_x = \alpha_u = \psi^* \alpha_{x'}$ and therefore

$$\sum_j \alpha_x^j = \sum_j \alpha_u^j = \sum \alpha_x^j, \quad \text{with } \alpha_*^j = \alpha_*^j(y_i).$$

Since X is connected it follows that $\alpha_x^1(y_i) \neq 0$ for all $x \in X$ and there exist $\xi_{ij}^u \in X(y_i, u)$ with the corresponding $\alpha_u^j(y_i) \neq 0$. Consequently there exist $\xi_{ij}^x \in X(y_i, x)$, $\xi_{ij}^{x'} \in X(y_i, x')$ with $\varphi \xi_{ij}^x = \xi_{ij}^u = \psi \xi_{ij}^{x'}$, i.e. the above diagram can be completed to



We have proved (ii)(2) and at the same time (ii)(1). We need only prove

(ii)(3). Let F_1 be the object of \mathbf{Ab}^X defined by

$$F_1(x) = \coprod_{\xi \in X(y, x)} Z\xi$$

with $y=y_i$ (i.e. the y_i picked above).

By (ii)(1) there exists an epimorphism $F_1 \rightarrow Z$ in \mathbf{Ab}^X . Since, by assumption, Z is projective as an object of \mathbf{Ab}^X , Z is a direct summand of F_1 , therefore Z is a direct summand of $F_1(y)$, as an $X(y, y)$ -module. But $F_1(y)$ can be identified with the monoid algebra $Z[X(y, y)]$ and it therefore follows that the cohomology of the monoid $M=X(y, y)$ is trivial.

LEMMA A. *If a monoid M is cohomologically trivial, then there exists an $\varepsilon \in M$ such that, for all $\xi \in M$, $\xi\varepsilon = \varepsilon$.*

PROOF. Consider the epimorphism $Z[M] \rightarrow Z$. Since cohomology is trivial, the corresponding homomorphism $H^0(M, Z(M)) \rightarrow H^0(M, Z) = Z$ is an epimorphism. Now $H^0(M, Z(M)) = \{ \sum_{i=1}^n \alpha_i \xi_i \mid \alpha_i \in Z, \xi_i \in M \text{ such that, for all } \xi \in M \text{ and all } i, \xi \xi_i = \xi_j \text{ for some } j=j_\xi \text{ with } \alpha_i = \alpha_{j_\xi} \}$.

It follows that there exists an element $\sum_{i=1}^n \alpha_i \xi_i \in Z(M)$ with $\sum_{i=1}^n \alpha_i = 1$ such that for all $\xi \in M$, $\xi \xi_i = \xi_{\sigma_\xi(i)}$ where σ_ξ is a permutation of $\{1, 2, \dots, n\}$.

Let $S(n)$ be the symmetric group, then the correspondence $\xi \rightarrow \sigma_\xi$ gives a homomorphism $\sigma: M \rightarrow S(n)$ since $\xi' \xi \xi_i = \xi' \xi_{\sigma_\xi(i)} = \xi_{\sigma_{\xi'}(\sigma_\xi(i))}$.

Let $H = \text{im } \sigma$ then H is a subgroup of $S(n)$. This follows from the fact that by the theorem of Lagrange, any submonoid of a finite group is a subgroup.

SUBLEMMA B. *If $M \xrightarrow{\sigma} H$ is a surjective homomorphism of monoids and if M is cohomologically trivial, then H is cohomologically trivial.*

PROOF. This follows from $H^0(M, -) = H^0(H, -)$ in the category of H -modules. Q.E.D.

SUBLEMMA C. *If a group G is cohomologically trivial, then $G = \{1\}$.*

PROOF. As above there exists an element $\sum_{i=1}^n \alpha_i \xi_i \in Z(G)$ such that for all $\xi \in G$, $\xi \xi_i = \xi_{\sigma_\xi(i)}$ and $\alpha_i = \alpha_{\sigma_\xi(i)}$, $\sum_{i=1}^n \alpha_i = 1$. Since for all i, j there exists an element ξ with $\sigma_\xi(i) = j$, we have $\alpha_i = \alpha_j$ for all i, j and $G = \{ \xi_1, \dots, \xi_n \}$. Since

$$1 = n \cdot \alpha_1 = |G| \cdot \alpha_1$$

it follows that $|G|=1$ and $\alpha_1=1$, and consequently $G = \{1\}$. Q.E.D.

Combining B and C we find that $\sigma_\xi = 1$ for all $\xi \in M$. This of course means that for all $\xi \in M$, $\xi \xi_i = \xi_i$. Put $\varepsilon = \xi_i$ for some i , and we have proved A. Q.E.D.

This ends the proof of the Theorem since A implies (ii)(3). Q.E.D.