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FINITE GROUP SCHEMES OVER FIELDS

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ABSTRACT. A short proof that commutative group schemes over a field form an abelian category is given.

The category of commutative finite group schemes (equivalently, the category of commutative, cocommutative Hopf algebras with antipode) over a field is abelian [3], [4, Chapter 16]. This note presents an elementary proof of this fact.

We make the following conventions: k will denote a field and k-Alg the category of commutative k-algebras; unsubscripted tensors are over k. Let k-FCGS denote the category of covariant functors from k-Alg to abelian groups represented by finite dimensional k-algebras. The objects of k-FCGS are called finite commutative group schemes. For G in k-FCGS, AG will denote the k-algebra representing G (also $G(T)=\mathrm{Hom}_{k\text{-Alg}}(AG, T)$). Let $[G:1]$ be the k-dimension of AG. By the Yoneda lemma [1, p. 113] (which we will subsequently employ without explicit reference), a morphism $f: G\to H$ in k-FCGS is induced by a unique morphism $Af: AH\to AG$ in k-Alg. A morphism $f: G\to H$ in k-FCGS is *free* if Af makes AG a free AH-module. We say "define $G\to H$ by $a\to a'$" for "let $G\to H$ be the transformation which, on T, sends $G(T)$ to $H(T)$ by sending a to a'".

We refer the reader to [3] for the facts that k-FCGS is a self-dual category ("Cartier duality"); that it has fibre products and cofibred coproducts, hence kernels and cokernels, and $A(G\times_H L)=AG\otimes_{AH} AL$; and that $G\to H$ is an epimorphism in k-FCGS if and only if $AH\to AG$ is injective. The null object 0 of k-FCGS is represented by k.

To show k-FCGS abelian, it suffices to prove, since the category is self dual, epimorphisms are cokernels. This will be done by showing that epimorphisms are free.

LEMMA 1. *Let $L\to G$ be the kernel of $G\to H$. Then $G\times_H G\to G$ (by projection on the first factor) is free of rank $[L:1]$.*

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PROOF. Since $L=G\times_H 0$, $L(T)\to G(T)$ is the kernel of $G(T)\to H(T)$. Define $d: G\times L\to G\times_H G$ by $(g, l)\to(g, gl)$. By the above, $d(T)$ is an isomorphism for each T, and hence d is an isomorphism. Moreover d commutes with projection on the first factor and hence defines an isomorphism of AG algebras $Ad: AG\otimes_{AH}AG\to AG\otimes AL$. The latter is free of rank $[L:1]$, whence the result.

LEMMA 2. *Let A and B be commutative rings, with A semilocal and B a finite faithful A-algebra. Then B is a free A-module of rank n if $B\otimes_A B$ is a free B-module* (*under left action*) *of rank n.*

PROOF. Let J be the Jacobson radical of A. We first suppose $J=0$. Then A is a finite product of fields, and in this case the result is immediate. In general, we observe that, by going up and going down, $JB\cap A=J$, so B/JB is faithful over A/J. Moreover, $B/JB\otimes_{A/J}B/JB=B/JB\otimes_B(B\otimes_A B)$ is free of rank n over B/JB, and so by the case already considered, B/JB is a free A/J-module of rank n. Choose $x_1, \cdots, x_n$ in B such that their images form a basis of B/JB over A/J. By Nakayama's lemma, the x_i span B. Moreover, the $1\otimes x_i$ then span $B\otimes_A B$; since the latter is free of rank n they are a basis. But any relation of linear dependence on the x_i over A gives a relation on the $1\otimes x_i$ over B, and hence must be trivial. Thus the x_i are free generators of B over A.

THEOREM. *Let $G\to H$ be an epimorphism in k-FCGS with kernel L. Then $G\to H$ is free of rank* $[L:1]$.

COROLLARY. *k-FCGS is an abelian category.*

PROOF. Let $p: G\to H$ be an epimorphism with kernel L. Let $C=$ Coker(ker(p)). By the definition of C, p factors as $G\to C\to H$ where both maps are epimorphisms. Since L is also the kernel of $G\to C$, the rank of AG as an AC module is $[L:1]$, which is the same as the rank of AG as an AH-module. Thus the inclusion $AH\to AC$ is an isomorphism and so $G\to H$ is a cokernel. Since the category is self dual, every monomorphism is a kernel.

(The corollary also follows from the theorem and the fact that faithfully flat morphisms are effective epimorphisms.)

We note that the theorem is a special case of the theorem that passage to the quotient by a finite flat equivalence relation on an affine scheme is faithfully flat [**2**, Theorem 4.1], and our approach is closely related to the proof of that theorem. Finally, it is the authors' belief that the mixture of cogroup and ring properties in the theory of Hopf algebras can make the subject difficult for nonspecialists; therefore it is often convenient to separate the properties (as we have tried to do here) by dealing with the functor of points, so ordinary group theory may be used as far as possible.

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