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PROPERTIES OF STONE-CECH COMPACTIFICATIONS OF DISCRETE SPACES[1]

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ABSTRACT. Let βN be the Stone-Čech compactification of the integers N. It is shown that if p is a P-point of $\beta N-N$, then $\beta N-N-\{p\}$ is not normal. Let D be an uncountable discrete set and E_0 be the set of points in $\beta D-D$ in the closures of countable subsets of D It is shown that there is a two-valued continuous function on E_0 which cannot be extended continuously to βD.

The purpose of this paper is to answer some questions raised in two papers, one by W. W. Comfort and S. Negrepontis [**1**] and the other by N. J. Fine and L. Gillman [**2**]. The first question, attributed to Gillman in [**1**], is whether $\beta N-N-\{p\}$ is normal when p is a P-point of $\beta N-N$. The answer is negative assuming the continuum hypothesis (designated by [CH]). The second question is raised in [**2**]. Let D be an uncountable discrete set and let E_0 be the set of points in $\beta D-D$ in the closures of countable subsets of D. Is E_0 C^*-embedded in $\beta D-D$? That is, does every bounded continuous function on E_0 have a continuous extension to $\beta D-D$? Again the answer is negative; in fact, there is a two-valued function on E_0 which cannot be extended continuously to $\beta D-D$.

Although the first question is a corollary to the second using a result from [**1**], I will sketch a straightforward proof of the answer.

I. THEOREM 1. [CH] *$\beta N-N-\{p\}$ is not normal if p is a P-point of $\beta N-N$.*

PROOF. Let $\{W_\alpha\}_{\alpha<\omega_1}$ be open-and-closed neighborhoods of p such that:

(i) $\{W_\alpha\}$ is a base at p;
(ii) $W_0=\beta N-N$;
(iii) $W_\gamma\subset W_\beta$ (properly) for $\beta<\gamma$;
(iv) $\bigcap_{\alpha<\omega_1} W_\alpha=\{p\}$.

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[1] This paper is based on my doctoral thesis written under the direction of Professor Mary Ellen Rudin to whom I wish to express my appreciation for her encouragement and guidance.

Let $U_\alpha = W_\alpha - W_{\alpha+1}$. Each U_α is open-and-closed in $\beta N - N$. Pick P-points $p_\alpha \in U_\alpha$. Then p is a limit point of $\{p_\alpha\}_{\alpha<\omega_1}$. For each $\alpha<\omega_1$, let $\{U_{\alpha\delta}\}_{\delta<\omega_1}$ be open-and-closed neighborhoods of p_α such that:

(i) $\{U_{\alpha\delta}\}$ is a base at p_α;

(ii) $U_{\alpha 0} \subset U_\alpha$;

(iii) $U_{\alpha\gamma} \subset U_{\alpha\beta}$ (properly) for $\beta<\gamma$;

(iv) $\bigcap_{\delta<\omega_1} U_{\alpha\delta} = \{p_\alpha\}$.

In $\beta N - N - \{p\}$, let $A = \mathrm{cl}(\{p_\alpha\})_{\alpha<\omega_1}$. Throughout the proof, λ will stand for a countable limit ordinal. Let

$$B_\lambda = \mathrm{cl}\left(\bigcup_{\alpha<\lambda} (U_\alpha - U_{\alpha\lambda})\right) \cap \left(\bigcap_{\alpha<\lambda} W_\alpha\right), \qquad \text{and let } B = \bigcup_{\lambda<\omega_1} B_\lambda.$$

I will show that A and B are disjoint closed sets in $\beta N - N - \{p\}$ which cannot be separated.

By definition, A is closed. To show that B is closed, suppose that q is a limit point of B and that β is the smallest ordinal such that $q \notin W_\beta$. I will show that if β is a nonlimit ordinal, then q cannot be a limit point of B. If β is a nonlimit ordinal, then $U_{\beta-1} = W_{\beta-1} - W_\beta$ is a neighborhood of q. If λ is a countable limit ordinal and $\lambda>\beta$, $B_\lambda \subset \bigcap_{\alpha<\lambda} W_\alpha \subset W_\beta$, so $B_\lambda \cap U_{\beta-1} = \varnothing$. If $\lambda<\beta$, $U_{\beta-1} \cap U_\alpha = \varnothing$ for $\alpha<\beta-1$, so

$$U_{\beta-1} \cap \left(\bigcup_{\alpha<\lambda} U_\alpha - U_{\alpha\lambda}\right) = \varnothing$$

and since $U_{\beta-1}$ is open, $U_{\beta-1} \cap \mathrm{cl}((\bigcup_{\alpha<\lambda} U_\alpha - U_{\alpha\lambda})) = \varnothing$. So $B_\lambda \cap U_{\beta-1} = \varnothing$ if $\lambda<\beta$. Now, since $B = \bigcup_{\lambda<\omega_1} B_\lambda$, $U_{\beta-1} \cap B = \varnothing$ so q is not a limit point of B.

If λ and λ' are distinct countable limit ordinals, then B_λ and $B_{\lambda'}$ are disjoint, and B_λ and $\mathrm{cl}(\{p_\alpha\})_{\alpha<\lambda}$ are disjoint, so A and B are disjoint. This relies on a lemma of M. E. Rudin [3, p. 148, Lemma 1].

To show that A and B cannot be separated, let $A \subset O$ and $B \subset V$ where O and V are open sets in $\beta N - N - \{p\}$. Then for each countable ordinal α, there exists a $\beta_\alpha > \alpha$ such that $U_{\alpha\beta_\alpha} \subset O$. Let $\alpha_0 = 0$, $\alpha_1 = \beta_{\alpha_0}$ and, in general, let $\alpha_n = \beta_{\alpha_{n-1}}$. Let γ be the limit of the sequence $\{\beta_{\alpha_n}\}$, then γ is also the limit of the sequence $\{\alpha_n\}$.

Now, $B_\gamma = \mathrm{cl}((\bigcup_{\alpha<\gamma} U_\alpha - U_{\alpha\gamma})) \cap (\bigcap_{\alpha<\gamma} W_\alpha)$ and $B_\gamma \subset V$. For each n, $U_{\alpha_n\beta_{\alpha_n}} - U_{\alpha_n\gamma} \subset O$ and $U_{\alpha_n\beta_{\alpha_n}} - U_{\alpha_n\gamma} \subset U_{\alpha_n} - U_{\alpha_n\gamma}$. So

$$V \cap \left[\bigcup_n (U_{\alpha_n\beta_{\alpha_n}} - U_{\alpha_n\gamma})\right] \neq \varnothing$$

and hence, V and O are not disjoint. So A and B cannot be separated in $\beta N - N - \{p\}$.

II. W. W. Comfort and S. Negrepontis have given in [1] a proof by L. Gillman that [CH] $\beta N-N-\{p\}$ is not normal if p is a non-P-point of $\beta N-N$. Also they prove that [CH] for each p in $\beta N-N$ there is a copy of $\beta N-N$ contained in $\beta N-N$ relative to which p is a P-point. Either of these results combined with Theorem 1 gives that [CH] *$\beta N-N-\{p\}$ is not normal if p is any point of $\beta N-N$.*

Let D_1 be a discrete set of cardinality $\aleph_1$ and let D_0 be the set of points in βD_1-D_1 in the closures of countable subsets of D_1. W. W. Comfort and S. Negrepontis show in [1] that [CH] if p is a P-point of $\beta N-N$, then D_0 is homeomorphic to $\beta N-N-\{p\}$. Observe that if we identify D_1 with any subset of D of cardinality $\aleph_1$, then D_0 is an open-and-closed subset of E_0. Hence, a consequence of Theorem 1 is that [CH] *E_0 is not normal.*

If D_0^* is the one-point compactification of D_0, then Comfort and Negrepontis have shown that [CH] D_0^* is homeomorphic to $\beta N-N$. Theorem 2, which does not use the continuum hypothesis, shows the existence of a continuous two-valued function on D_0 which cannot be extended continuously to βD_1-D_1. Such a function cannot be extended continuously to D_0^*. So Theorem 2 implies Theorem 1. In fact, a stronger theorem which does not use the continuum hypothesis is true. *If D_0^* is homeomorphic to $\beta N-N$ and $p\in\beta N-N$, then $\beta N-N-\{p\}$ is not normal.* Perhaps, the reason one so often needs the continuum hypothesis to prove theorems about $\beta N-N$ is that one wants $\beta N-N$ to be homeomorphic to D_0^*. Certainly this substitution can often be made and it would be interesting to find out how often.

III. THEOREM 2. *There is a continuous two-valued function on E_0 which cannot be continuously extended to $\beta D-D$.*

PROOF. Without loss of generality assume that the cardinality of D is $\aleph_1$. This can be done since, as was pointed out in §II, D_0 is both open and closed in E_0.

N. Aronszajn [4] and F. B. Jones [5] have shown the existence of a partial order $\leqq$ on D such that:

(i) If $x\in D$, $\{y\in D: y\leqq x\}$ is well ordered by $\leqq$.

(ii) Every totally ordered subset of D is countable.

(iii) For each countable ordinal α, the set $X_\alpha=\{x\in D:\{y\in D: y\leqq x\}$ is order isomorphic with $\alpha\}$ is countable.

(iv) $D=\bigcup_{\alpha<\omega_1} X_\alpha$.

Let $Y_\alpha=\bigcup_{\beta<\alpha} X_\beta$. For each countable ordinal α, I will select $Z_\alpha\subset Y_\alpha$ by induction. Let $Z_0=\varnothing$. If α is not a limit ordinal, $\alpha\geqq 1$, let $Z_\alpha=Z_{\alpha-1}$.

If α is a limit ordinal, index the terms of X_α by the integers as $\{x_n^\alpha\}$. Select a sequence of ordinals, $\{\beta_n\}$, $0=\beta_0<\beta_1<\beta_2<\cdots$, having α as a limit. Let $p_n^{\beta_n}$ be the term of X_{β_n} such that $p_n^{\beta_n}<x_n^\alpha$. Suppose that Z_β has

been defined for all $\beta<\alpha$, then

$$Z_\alpha = \bigcup_{n\geqq 1}(Z_{\beta_n} - Y_{\beta_{n-1}}) \cup \bigcup_{n\geqq 1}\{x : p_n^{\beta_n} \leqq x < x_n^\alpha\}.$$

If $A\subset D$, let $\bar{A}$ denote the intersection of $\beta D-D$ with the closure of A in βD. Then $E_0=\bigcup_{\alpha<\omega_1}\bar{Y}_\alpha$ and $\bar{Y}_\alpha=\bar{Z}_\alpha\cup\mathrm{cl}((Y_\alpha-Z_\alpha))$ so

$$E_0 = \left(\bigcup_{\alpha<\omega_1} \bar{Z}_\alpha\right) \cup \left(\bigcup_{\alpha<\omega_1} \mathrm{cl}((Y_\alpha - Z_\alpha))\right). \tag{*}$$

I will show that $\bigcup_{\alpha<\omega_1}\bar{Z}_\alpha$ and $\bigcup_{\alpha<\omega_1}\mathrm{cl}((Y_\alpha-Z_\alpha))$ are disjoint, closed sets in E_0 which cannot be separated in $\beta D-D$.

First, I will prove two lemmas to establish some properties of the Z_α's.

Lemma 1. *For all α and β, countable ordinals, $X_\alpha\cap Z_\beta$ is finite.*

The proof is by induction on β.

If $\beta=0$, $Z_0=\varnothing$, so $X_\alpha\cap Z_0=\varnothing$.

Assume that for all $\gamma<\beta$, $X_\alpha\cap Z_\gamma$ is finite for all α. If β is a nonlimit ordinal, $Z_\beta=Z_{\beta-1}$ and $X_\alpha\cap Z_{\beta-1}$ is finite by the induction hypothesis.

If β is a limit ordinal, then

$$Z_\beta = \bigcup_{n\geqq 1}(Z_{\beta_n} - Y_{\beta_{n-1}}) \cup \bigcup_{n\geqq 1}\{x : p_n^{\beta_n} \leqq x < x_n^\beta\}$$

where $0=\beta_0<\beta_1<\cdots$ is a previously chosen sequence converging to β. If $\alpha\geqq\beta$, $X_\alpha\cap Z_\beta=\varnothing$. Otherwise, there exists an integer i such that $\beta_i\leqq\alpha<\beta_{i+1}$. Then $X_\alpha\subset Y_{\beta_{i+1}}$. So

$$X_\alpha \cap Z_\beta = X_\alpha \cap \left[\bigcup_{n=1}^{i+1}(Z_{\beta_n} - Y_{\beta_{n-1}}) \cup \bigcup_{n=1}^{i}\{x : p_n^{\beta_n} \leqq x < x_n^\beta\}\right].$$

$X_\alpha\cap Z_{\beta_n}$ is finite by the induction hypothesis for $n=1,\cdots,i+1$. Each set of the form $\{x:p_n^{\beta_n}\leqq x<x_n^\beta\}$ is totally ordered, so $X_\alpha\cap\{x:p_n^{\beta_n}\leqq x<x_n^\beta\}$ contains at most one element for each $n=1,\cdots,i$. So $X_\alpha\cap Z_\beta$ is finite.

Lemma 2. *If β and α are countable ordinals and $\beta<\alpha$, then $Z_\beta-Z_\alpha$ and $(Y_\beta-Z_\beta)\cap Z_\alpha$ are finite.*

The proof is by induction on α.

If $\alpha=1$ and $\beta=0$, then $Z_0=\varnothing$ and $Z_1=\varnothing$ so $Z_0-Z_1=\varnothing$ and $(Y_0-Z_0)\cap Z_1=\varnothing$.

Assume that for all $\gamma<\alpha$, both $Z_\beta-Z_\gamma$ and $(Y_\beta-Z_\beta)\cap Z_\gamma$ are finite for all $\beta<\gamma$.

If α is a nonlimit ordinal, $Z_\alpha=Z_{\alpha-1}$ and by the induction hypothesis, for all $\beta<\alpha-1$, both $Z_\beta-Z_{\alpha-1}$ and $(Y_\beta-Z_\beta)\cap Z_{\alpha-1}$ are finite and, clearly, $Z_{\alpha-1}-Z_\alpha=\varnothing$ and $(Y_{\alpha-1}-Z_{\alpha-1})\cap Z_\alpha=\varnothing$.

If α is a limit ordinal, let β' be the smallest ordinal less than α for which it is not known that both $Z_{\beta'}-Z_\alpha$ and $(Y_{\beta'}-Z_{\beta'})\cap Z_\alpha$ are finite.

If β' is a nonlimit ordinal, then $Z_{\beta'}=Z_{\beta'-1}$ and both $Z_{\beta'-1}-Z_\alpha$ and $(Y_{\beta'-1}-Z_{\beta'-1})\cap Z_\alpha$ are finite by the choice of β'.

Suppose β' is a limit ordinal. By definition

$$Z_\alpha = \bigcup_{n\geqq 1}(Z_{\beta_n} - Y_{\beta_{n-1}}) \cup \bigcup_{n\geqq 1}\{x : p_n^{\beta_n} \leqq x < x_n^\alpha\}$$

for a previously chosen sequence $0=\beta_0<\beta_1<\cdots$ converging to α. There exists an i such that $\beta_i\leqq\beta'<\beta_{i+1}$ and then $Z_{\beta'}\subset Y_{\beta_{i+1}}$ and $Y_{\beta'}-Z_{\beta'}\subset Y_{\beta_{i+1}}$. So

$$Z_{\beta'} - Z_\alpha = \bigcup_{n=1}^{i+1}[[Z_{\beta'} \cap (Y_{\beta_n} - Y_{\beta_{n-1}})] - Z_\alpha].$$

For $n=1,\cdots,i$,

$$\begin{aligned}[Z_{\beta'} \cap (Y_{\beta_n} - Y_{\beta_{n-1}})] - Z_\alpha &\subset [Z_{\beta'} \cap (Y_{\beta_n} - Y_{\beta_{n-1}})] - (Z_{\beta_n} - Y_{\beta_{n-1}})\\ &\subset (Y_{\beta_n} - Z_{\beta_n}) \cap Z_{\beta'}\end{aligned}$$

which is finite by the induction hypothesis. Also,

$$\begin{aligned}[Z_{\beta'} \cap (Y_{\beta_{i+1}} - Y_{\beta_i})] - Z_\alpha &\subset [Z_{\beta'} \cap (Y_{\beta_{i+1}} - Y_{\beta_i})] - [Z_{\beta_{i+1}} - Y_{\beta_i}]\\ &\subset Z_{\beta'} - Z_{\beta_{i+1}}\end{aligned}$$

which is finite by the induction hypothesis. So $Z_{\beta'}-Z_\alpha$ is finite.

Also,

$$\begin{aligned}&(Y_{\beta'} - Z_{\beta'}) \cap Z_\alpha\\ &\qquad = (Y_{\beta'} - Z_{\beta'}) \cap \left[\bigcup_{n=1}^{i+1}(Z_{\beta_n} - Y_{\beta_{n-1}}) \cup \bigcup_{n=1}^{i}\{x : p_n^{\beta_n} \leqq x < x_n^\alpha\}\right].\end{aligned}$$

For $n=1,\cdots i$,

$$(Y_{\beta'} - Z_{\beta'}) \cap (Z_{\beta_n} - Y_{\beta_{n-1}}) \subset Z_{\beta_n} - Z_{\beta'},$$

and

$$(Y_{\beta'} - Z_{\beta'}) \cap (Z_{\beta_{i+1}} - Y_{\beta_i}) \subset (Y_{\beta'} - Z_{\beta'}) \cap Z_{\beta_{i+1}},$$

both of which are finite by the induction hypothesis.

Now let $1\leqq n\leqq i$ and consider $(Y_{\beta'}-Z_{\beta'})\cap\{x:p_n^{\beta_n}\leqq x<x_n^\alpha\}$. There is a point $y\in X_{\beta'}$ such that $p_n^{\beta_n}<y<x_n^\alpha$. Next, by definition of $Z_{\beta'}$, since β' is a limit ordinal, there are an ordinal $\gamma<\beta'$ and a point $z\in X_\gamma$ such that $Z_{\beta'}\cap\{x:x<y\}=\{x:z<x<y\}$. Then

$$(Y_{\beta'} - Z_{\beta'}) \cap \{x : p_n^{\beta_n} \leqq x < x_n^\alpha\} = \{x : p_n^{\beta_n} \leqq x < z\}$$

$(=\varnothing$ in case $z\leqq p_n^{\beta_n}$, i.e., $\gamma\leqq\beta_n)$. Since $\{x: p_n^{\beta_n}\leqq x<z\}\subset Y_\gamma$ we have

$$(Y_{\beta'} - Z_{\beta'}) \cap \{x : p_n^{\beta_n} \leqq x < x_n^\alpha\} \subset [(Y_\gamma - Z_\gamma) \cup (Z_\gamma - Z_{\beta'})] \cap Z_\alpha,$$

which is finite, since by the induction hypotheses, both $(Y_\gamma - Z_\gamma)\cap Z_\alpha$ and $Z_\gamma - Z_{\beta'}$ are finite. It follows that $(Y_{\beta'} - Z_{\beta'})\cap Z_\alpha$ is finite.

PROOF OF THEOREM 2. To show that $\bigcup_{\alpha<\omega_1} \bar{Z}_\alpha$ and $\bigcup_{\alpha<\omega_1} \mathrm{cl}((Y_\alpha - Z_\alpha))$ are closed and disjoint in E_0, consider $\bar{Z}_\gamma$ and $\mathrm{cl}(Y_\beta - Z_\beta)$ for γ, $\beta<\omega_1$. If $\gamma=\beta$, then $Z_\gamma\cap(Y_\gamma - Z_\gamma)=\varnothing$ so $\mathrm{cl}_{\beta D} Z_\gamma \cap \mathrm{cl}_{\beta D}(Y_\gamma - Z_\gamma)=\varnothing$ [6], and therefore, $\bar{Z}_\gamma$ and $\mathrm{cl}(Y_\gamma - Z_\gamma)$ are disjoint. If $\gamma<\beta$, $Z_\gamma\subset Y_\beta$ and $Z_\gamma - Z_\beta$ is finite by Lemma 2. So $Z_\gamma\cap(Y_\beta - Z_\beta)$ is finite, and hence $\bar{Z}_\gamma\cap \mathrm{cl}(Y_\beta - Z_\beta)=\varnothing$. If $\beta<\gamma$, then $(Y_\beta - Z_\beta)\cap Z_\gamma$ is finite by Lemma 2, so $\mathrm{cl}((Y_\beta - Z_\beta))\cap \bar{Z}_\gamma=\varnothing$. So $\bigcup_{\alpha<\omega_1} \bar{Z}_\alpha$ and $\bigcup_{\alpha<\omega_1} \mathrm{cl}((Y_\alpha - Z_\alpha))$ are disjoint.

Next, these sets are open, since any set of the form $\bar{A}$ is open. Finally, since they are complementary (*), they are also closed.

If $\bigcup_{\alpha<\omega_1} \bar{Z}_\alpha$ and $\bigcup_{\alpha<\omega_1} \mathrm{cl}((Y_\alpha - Z_\alpha))$ can be separated in $\beta D - D$, then there exists a set $Z\subset D$ such that $\bar{Z}\supset \bigcup_{\alpha<\omega_1} \bar{Z}_\alpha$ and

$$\bar{Z}\cap\left(\bigcup_{\alpha<\omega_1} \mathrm{cl}((Y_\alpha - Z_\alpha))\right) = \varnothing .$$

Then $\bar{Z}_\alpha\subset \bar{Z}$ and so $Z_\alpha - Z$ must be finite for all $\alpha<\omega_1$. Also if $\bar{Z}\cap \mathrm{cl}((Y_\alpha - Z_\alpha))=\varnothing$ then $Z\cap(Y_\alpha - Z_\alpha)$ is finite. Since $Y_\alpha=\bigcup_{\beta<\alpha} X_\beta$, $Z\cap(X_\beta - Z_\alpha)$ must, for all $\alpha<\omega_1$, be finite for all $\beta<\alpha$. But since $X_\beta\cap Z_\alpha$ is finite by Lemma 1, $Z\cap X_\beta$ must be finite for all $\beta<\omega_1$.

I will complete the proof by showing that there is no set $Z\subset D$ with the properties described in the preceding paragraph.

Suppose there is such a set Z. Then for every countable limit ordinal α, since $Z_\alpha - Z$ is finite, there exists an $f(\alpha)<\alpha$ such that if y is the term of $X_{f(\alpha)}$ such that $y<x_1^\alpha$, then $\{x: y\leqq x<x_1^\alpha\}\subset Z$. If α is a nonlimit ordinal, let $f(\alpha)=\alpha-1$ and let $f(0)=0$.

Then there exists $\gamma<\omega_1$ such that $f(\alpha)=\gamma$ for uncountably many α. For, if not, then $\{\alpha: f(\alpha)\leqq\beta\}$ is countable for each $\beta<\omega_1$, and letting $C_0=\{\alpha: f(\alpha)=0\}$; C_0 is countable and has a supremum α_1, $\alpha_1>0$. In general, let $C_n=\{\alpha: f(\alpha)\leqq\alpha_n\}$; C_n is countable and has a supremum α_{n+1}, where $\alpha_{n+1}>\alpha_n$, since α_n+1 is in C_n. Then if α' is the limit of the sequence $\{\alpha_n\}$, and since $f(\alpha')<\alpha'$, $f(\alpha')\leqq\alpha_i$ for some i, but then α' is in C_i and $\alpha'\leqq\alpha_{i+1}$ which is a contradiction of the choice of α'.

Let q be a point of X such that $q<x_1^\alpha$ for uncountably many α's such that $\gamma=f(\alpha)$. Let $W=\{x: q<x$ and $\{y: q<y\leqq x\}\subset Z\}$. Then W is uncountable by the choice of q. Let $A=\{x: q<x$ and there are uncountably many $z>x$ such that $\{y: q<y\leqq z\}\subset Z\}$. Observe that $A\subset W$. Also A is uncountable. Because if A is countable, then $A\subset Y_\beta$ for some $\beta>\gamma$, and if $x'\in X_\beta$, there are only countably many $z>x'$ such that $\{y: q<y\leqq z\}\subset Z$. So $W-Y_\beta$ is countable, hence W is countable which is a contradiction.

For each $\alpha>\gamma$, let $n(\alpha)=$the number of terms in $A\cap X_\alpha$. Then $n(\alpha)$ is finite since $A \subset Z$ and $n(\alpha) > 0$ since A is uncountable. Pick n_0 so that for uncountably many $\alpha > \gamma$, $n(\alpha) = n_0$. Let α_0 be the first ordinal greater than γ such that $n(\alpha_0) = n_0$. Then for each ordinal $\alpha > \alpha_0$ such that $n(\alpha) = n_0$, define $F_\alpha : X_\alpha \cap A \to X_{\alpha_0} \cap A$ where $F_\alpha(y)$ is the term of X_{α_0} between q and y. Clearly $F_\alpha(y) < y$ for all α. If $y \in A$, there are uncountably many $z > y$ such that $\{x : q < x \leqq z\} \subset Z$ and since $F_\alpha(y) < y$, there are uncountably many $z > F_\alpha(y)$ such that $\{x : q < x \leqq z\} \subset Z$, so $F_\alpha(y) \in A$.

I will now show that F_α is one-to-one. If $x \in X_{\alpha_0} \cap A$, there is a $z > x$, $z \notin Y_\alpha$ and $z \in A$. Then the term y of X_α between q and z belongs to A and $F_\alpha(y) = x$. So, since $X_\alpha \cap A$ and $X_{\alpha_0} \cap A$ each have n_0 terms, F_α is one-to-one.

Let $x_0 \in X_{\alpha_0} \cap A$. For each $\alpha > \alpha_0$ with $n(\alpha) = n_0$, let y_α be the unique term of X_α such that $x_0 = F_\alpha(y_\alpha)$. Then $B = \{y_\alpha\}$ is a totally ordered subset of D, since if $\alpha_1 < \alpha_2$, then there is a term y of X_{α_1} between x_0 and y_{α_2}. But $F_\alpha(y) = x_0$; hence $y = y_{\alpha_1}$; so $y_{\alpha_1} < y_{\alpha_2}$. But since there are uncountably many α such that $n(\alpha) = n_0$, B is uncountable which is a contradiction.

So there is no set Z in D which separates

$$\bigcup_{\alpha < \omega_1} Z_\alpha \quad \text{and} \quad \bigcup_{\alpha < \omega_1} \mathrm{cl}((Y_\alpha - Z_\alpha))$$

in $\beta D - D$.

So if f is a function from E_0 to $\{0, 1\}$ such that $f^{-1}(0) = \bigcup_{\alpha < \omega_1} Z_\alpha$ and $f^{-1}(1) = \bigcup_{\alpha < \omega_1} \mathrm{cl}((Y_\alpha - Z_\alpha))$, then f is continuous on E_0, but f has no continuous extension to $\beta D - D$.

COROLLARY. *E_0 is not normal.*

PROOF. For each limit ordinal $\lambda < \omega_1$, let $Q_\lambda = Y_{\lambda + \omega_0} - Y_\lambda$. Let $A = \bigcup_{\lambda < \omega_1} \mathrm{cl}_{\beta D} Q_\lambda$, then $D \subset A \subset \beta D$ so $\beta A = \beta D$. Pick $p_\lambda \in \bar{Q}_\lambda$. Then $D' = \{p_\lambda\}_{\lambda < \omega_1}$ is a discrete set in $\beta D - D$ of cardinality $\aleph_1$ and $D' \subset E_0$. D' is C^*-embedded in A, since to extend a function f from D' to A, assign the value $f(p_\lambda)$ to each point of $\mathrm{cl}_{\beta D} Q_\lambda$. Therefore, $\mathrm{cl}_{\beta A} D' = \mathrm{cl}_{\beta D} D' = \beta D'$, and since $D' \subset \beta D - D$, so is $\beta D'$.

Let E_0' be the set of all points in $\beta D'$ which are limit points of countable subsets of D'. Then E_0' is a closed subset of E_0. Applying Theorem 2 to E_0', there is a continuous function g from E_0' to $\{0, 1\}$ which cannot be extended continuously to $\beta D' - D'$. Then g cannot be extended continuously to $E_0' \cup D'$ since $\beta(E_0' \cup D') = \beta D'$. Now, $g^{-1}(0)$ and $g^{-1}(1)$ are disjoint closed sets in E_0. Suppose U and V are open in E_0 and $g^{-1}(0) \subset U$, $g^{-1}(1) \subset V$ and $U \cap V = \varnothing$. Then all but at most a finite number of elements of D' are in $U \cup V$, so g can be extended continuously to $E_0' \cup D'$, which is a contradiction to the choice of g, so E_0 is not normal.

COROLLARY. *$E_0 \cup D$ is not normal.*

PROOF. Since E_0 is closed in $E_0 \cup D$, $E_0 \cup D$ is not normal.

References

1. W. W. Comfort and S. Negrepontis, *Homeomorphs of three subspaces of $\beta N/N$*, Math. Z. **107** (1968), 53–58. MR **38** #2739.

2. N. J. Fine and L. Gillman, *Extension of continuous functions in βN*, Bull. Amer. Math. Soc. **66** (1960), 376–381. MR **23** #A619.

3. M. E. Rudin, *Types of ultrafilters*, Topology Seminar (Wisconsin, 1965), Ann. of Math. Studies, no. 60, Princeton Univ. Press, Princeton, N.J., 1966, p. 148. MR **35** #7284.

4. G. Kurepa, *Ensembles linéaires et une classe de tableaux ramifiés* (tableaux ramifiés de M. Aronszajn), Publ. Math. Univ. Belgrade **6** (1937), 129–160.

5. F. B. Jones, *On certain well-ordered monotone collections of sets*, J. Elisha Mitchell Sci. Soc. **69** (1953), 33–34. MR **15**, 18; 1139.

6. L. Gillman and M. Jerison, *Rings of continuous functions*, The University Series in Higher Math., Van Nostrand, Princeton, N.J., 1960, p. 86. MR **22** #6994.

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