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A PROPERTY OF $y''' + p(x)y' + \frac{1}{2}p'(x)y = 0$

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ABSTRACT. It is shown that if $y'''+p(x)y'+\frac{1}{2}p'(x)y=0$ has an oscillatory solution, it has bases with 0, 1, 2, or 3 oscillatory elements.

It has recently been observed by Utz [3] that it is possible for a third order differential equation

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0 \tag{1}$$

to have bases for its solution space consisting of 0, 1, 2, or 3 oscillatory elements, where to say a solution of (1) is oscillatory means it changes sign for arbitrarily large x. The example he gave is

$$y''' - 3y'' + 4y' - 2y = 0, \tag{2}$$

which has solutions $y_1=e^x \sin x$, $y_2=e^x \cos x$, $y_3=e^x$. However, there is no loss in generality in assuming $p(x)\equiv 0$ in (1) in investigating oscillating solutions, for $p(x)$ can be eliminated from (1) by the transformation

$$y = u \exp\left(-\frac{1}{3}\int_0^x p(t)\, dt\right).$$

Applying this transformation, (2) becomes

$$y''' + y' = 0. \tag{3}$$

In an equation of the form

$$y''' + q(x)y' + r(x)y = 0 \tag{4}$$

where $q(x)$ and $r(x)$ are constant, it is not difficult to show that it is possible for (4) to have bases for its solution space consisting of 0, 1, 2, or 3 oscillating elements if and only if $r=0$. But, this is equivalent, when $q(x)$ and $r(x)$ are constant, to saying that (4) is selfadjoint.

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We now will prove the following generalization of the constant coefficient case for the general third order selfadjoint equation

$$y''' + b(x)y' + \tfrac{1}{2}b'(x)y = 0, \tag{5}$$

where $b(x)$ and $b'(x)$ are assumed to be continuous on $(0, +\infty)$.

THEOREM. *If* (5) *has an oscillatory solution then its solution space has bases consisting of* 0, 1, 2, *or* 3 *oscillatory elements.*

PROOF. It is well known [**1**] that the general solution of (5) is

$$y(x) = k_1 y_1^2(x) + k_2 y_1(x) y_2(x) + k_3 y_2^2(x),$$

where $y_1(x)$ and $y_2(x)$ are solutions of

$$y'' + \tfrac{1}{4}b(x)y = 0, \tag{6}$$

and k_1, k_2, k_3 are arbitrary constants.

If $u(x)$ and $v(x)$ are linearly independent solutions of (6), between two consecutive zeros of $u(x)$ there will be exactly one zero of $v(x)$ [**2**, p. 177]. Thus it is clear that $u(x)$ is oscillatory if and only if $v(x)$ is oscillatory, and that $u(x)$ and $v(x)$ cannot have a zero in common. From this it follows that if (6) is oscillatory, then $u(x)v(x)$ is an oscillatory solution of (5), where $u(x)$ and $v(x)$ are linearly independent solutions of (6). Using these facts, if (6) is oscillatory, it is easy to verify that if $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (6), the following four bases of (5) have 3, 2, 1, and 0 oscillatory elements respectively:

$$y_1(x)y_2(x), \quad y_1(x)(y_1(x) + y_2(x)), \quad y_2(x)(y_1(x) + y_2(x));$$
$$y_1(x)y_2(x), \quad y_1(x)(y_1(x) + y_2(x)), \quad y_1^2(x) + y_2^2(x);$$
$$y_1(x)y_2(x), \quad y_1^2(x) + 2y_2^2(x), \quad 2y_1^2(x) + y_2^2(x);$$
$$y_1^2(x) + 2y_2^2(x), \quad 2y_1^2(x) + y_2^2(x), \quad (y_1(x) + y_2(x))^2 + y_1^2(x).$$

It remains to show that if (5) has an oscillatory solution, then (6) is oscillatory. Suppose $y_1(x)$ and $y_2(x)$ are chosen so that

$$y_1(1) = 0 = y_2'(1), \qquad y_1'(1) = 1 = y_2(1).$$

Suppose $y_1(x)$ is not oscillatory. Then

$$y(x) = y_1^2(x) + k_1 y_1(x) y_2(x) + k_2 y_2^2(x)$$

is oscillatory for some choice of k_1 and k_2 not both zero. But a theorem of Birkhoff [**1**] says that if $u(x)$ and $v(x)$ are linearly independent solutions of (5) with at least one zero, then the zeros of $u(x)$ and $v(x)$ separate each other singly or in pairs. Thus since $y_1^2(1)=0$ it must have an infinity of zeros. Since $y_1(x)$ cannot have a double zero it must be oscillatory.

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