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UNITARY BORDISM OF ABELIAN GROUPS

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ABSTRACT. It is shown that for a finite abelian group G the bordism group of unitary G-manifolds is a free U_*-module on even dimensional generators.

Introduction. In this note we study from the bordism point of view actions of a finite abelian group G on unitary manifolds such that the action commutes with the stable complex structure on the tangent bundle. The corresponding bordism group $U_*(G)$ is a graded module over the complex bordism ring U_*.

In [4] Stong had shown that for a finite p-primary group G the module $U_*(G)$ is free on even dimensional generators. Landweber [3] proved the same result for finite cyclic G.

Using Stong's result we give a simple proof of the following.

THEOREM 1. *Let G be a finite abelian group. Then $U_*(G)$ is a free U_*-module on even dimensional generators.*

In [2] it was shown how such results for finite groups imply similar ones for higher dimensional groups. In fact, from the knowledge of Theorem 1 the methods of [2] give immediately

THEOREM 2. *Let G be a compact abelian Lie group and let $U_*^{(j)}(G)$ be the bordism module of actions such that each isotropy subgroup has codimension at least j. Then $U_*^{(j)}(G)$ is a free U_*-module on generators in dimension congruent to j modulo 2.*

In [2] this theorem was proved for a monogenic group G, using Landweber's result instead of Theorem 1.

Throughout the paper G will be a finite abelian group. The order of G is denoted by $|G|$.

1. **Preliminaries.** We shall have to consider also bordism modules of G-vectorbundles. The most general bordism group arising in this connection is defined as follows:

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Let $k_1, \cdots, k_m$ be nonnegative integers and let $F, F_1, \cdots, F_m$ be families of subgroups of G. Consider complex G-vectorbundles $E_1, \cdots, E_m$ over closed unitary G-manifolds M (in the sense of the Introduction) with the following properties:

(i) All isotropy groups in M are elements of F,

(ii) E_i is of dimension k_i and all isotropy groups in the sphere bundle of E_i are elements of F_i.

The corresponding bordism group of such objects $(M, E_1, \cdots, E_m)$ will be denoted by $U_*(G; F; (k_1, F_1), \cdots, (k_m, F_m))$. If any of the families $F, F_1, \cdots, F_m$ is the family of all subgroups of G we delete it from the notation; moreover, we will simply write (k) instead of $k_1, \cdots, k_m$.

We shall actually prove

THEOREM 1′. *Let G be a finite abelian group. Then $U_*(G; (k))$ is a free U_*-module on even dimensional generators.*

This induces a similar generalization of Theorem 2.

Our basic tool is the well-known exact sequence introduced by Conner [1]. Let F be as above and let H be an element of F. Define $F_H = \{H' \in F \mid H \subset H'\}$ and $F' = F - F_H$. Then the following sequence of U_*-modules is exact:

$$\begin{array}{ccc} U_*(G; F'; (k)) & \xrightarrow{i} & U_*(G; F; (k)) \\ \nwarrow \partial & & \downarrow j \\ & \bigoplus_{k_0 > 0} U_*(G; F_H; (k_0, F'), (k)) & \end{array}$$

Here i is the obvious forgetful map,

$$j(M, E_1, \cdots, E_m) = \sum_i (M_i^H, \nu_i, E_1 \mid M_i^H, \cdots, E_m \mid M_i^H)$$

where M_i^H is a component of the fixed point set of H in M and ν_i its normal bundle, and finally $\partial(M, E_0, E_1, \cdots, E_m) = (S, \pi^* E_1, \cdots, \pi^* E_m)$ where $\pi: S \to M$ is the projection of the sphere bundle S of E_0. The exactness follows from obvious geometric arguments.

For an element $(M, E_0, E_1, \cdots, E_m)$ in $U_*(G; F_H; (k_0, F'), (k))$ the action of H is just given by representations of H. It is possible to describe the G-action by means of these representations and a G/H-action. Then in many cases $U_*(G; F_H; (k_0, F'), (k))$ is isomorphic to a direct sum of bordism modules $U_*(G/H; \tilde{F}; (\tilde{k}))$ (see for instance [2, proof of Lemma 1] where a similar situation is studied in detail). Later on we shall use such isomorphisms without further notice.

2. **Proof of Theorem 1′.** Write the finite abelian group G in the form $G=K\times L$ where the orders $|K|$ and $|L|$ are relatively prime. Let $d=|K|$ and let R_d be the ring $R_d=\boldsymbol{Z}[1/d]\subset \boldsymbol{Q}$.

PROPOSITION 2. *Let F be a family of subgroups of L and let f: $U_*(K\times L;\, F;\, (k))\to U_*(L;\, F;\, (k))$ be defined by forgetting the action of K. Then $f\otimes\mathrm{id}\colon U_*(K\times L;\, F;\, (k))\otimes R_d\to U_*(L;\, F;\, (k))\otimes R_d$ is an isomorphism.*

REMARK. The inverse isomorphism is then given by multiplication with $[K]\otimes 1/d$ where $[K]$ is the bordism class of the free K-manifold K.

PROOF OF PROPOSITION 2. We proceed by induction over L and the family F.

If $F=\{\{1\}\}$ contains only the trivial group we have $U_*(K\times L;\, F;\, (k))\cong U_*(BK\times BL\times\prod BU(k_i))$ where the right-hand side is the usual bordism group of the classifying space $B(K\times L\times \prod U(k_i))$. But since $\tilde{H}_*(BK;\, R_d)=0$ it follows that

$$\begin{aligned} U_*\big(BK\times BL\times \textstyle\prod BU(k_i)\big)\otimes R_d &\cong U_*\big(BL\times \textstyle\prod BU(k_i)\big)\otimes R_d\\ &\cong U_*(L;\, F;\, (k))\otimes R_d \end{aligned}$$

and this isomorphism is given by $x\mapsto f(x)\otimes 1/d$. Hence $f\otimes\mathrm{id}$ is an isomorphism for $F=\{\{1\}\}$.

Now let F be arbitrary and H a maximal element of F. Set $F'=F-\{H\}$. Then one has the following commutative diagram of exact sequences:

$$\begin{tikzcd} U_*(K\times L;\, F';\, (k)) \arrow[r, "i"] \arrow[d, "f"] & U_*(K\times L;\, F;\, (k)) \arrow[r, "j"] \arrow[d, "f"] & \bigoplus U_*(K\times L/H;\, \{\{1\}\};\, (\tilde{k})) \arrow[d, "f"] \arrow[ll, bend right, "\partial"'] \\ U_*(L;\, F';((k)) \arrow[r, "i"] & U_*(L;\, F,\, (k)) \arrow[r, "j"] & \bigoplus (U_*(L/H;\, \{\{1\}\};\, (\tilde{k})) \arrow[ll, bend left, "\partial"] \end{tikzcd}$$

Tensoring with R_d, the induction hypothesis, and the 5-lemma give then the desired result.

PROPOSITION 3. *Assume that $U_*(L;\, (k))\otimes R_d$ is a free $U_*\otimes R_d$-module on even dimensional generators for all (k). Let F be a family of subgroups of K, Φ the family of all subgroups of L and $F\times\Phi=\{H\times H'\mid H\in F,\ H'\in\phi\}$. Then $U_*(K\times L;\, F\times\Phi;\, (k))\otimes R_d$ is a free $U_*\otimes R_d$-module on even dimensional generators.*

PROOF. We proceed by induction over K and F. The case $F=\{\{1\}\}$ is just Proposition 2 together with the assumption.

Let H be a maximal element of F and $F'=F-\{H\}$. Then we have the exact sequence

$$\begin{tikzcd} U_*(K \times L; F' \times \phi; (k)) \arrow[rr, "i"] & & U_*(K \times L; F \times \phi; (k)) \arrow[dl, "j"] \\ & \oplus U_*(K/H \times L; \phi; (\tilde{k})) \arrow[ul, "\partial"] & \end{tikzcd}$$

Tensoring with R_d, applying the induction hypotheses, and separating even and odd dimensions (i and j preserve dimensions, ∂ lowers dimensions by 1) yields the result.

COROLLARY 4. *If $U_*(L; (k))\otimes R_d$ is a free $U_*\otimes R_d$-module on even dimensional generators for all (k), then the same statement holds for $U_*(K\times L; (k))\otimes R_d$.*

PROOF. Since $|K|$ and $|L|$ are relatively prime, any subgroup of $K\times L$ is a direct product of subgroups of K and L.

Now we are able to prove Theorem 1′ by induction over the number of primary factors of G. For G p-primary the result was proved in [**4**]. For arbitrary G the induction hypothesis and Corollary 4 show that for any integer $d>1$ dividing $|G|$ the $U_*\otimes R_d$-module $U_*(G; (k))\otimes R_d$ is free on even dimensional generators. The assertion follows now from the trivial:

LEMMA 5. *Let M be a graded U_*-module of finite type (that is $M/\oplus_{n\geq N} M^n$ is finitely generated for any N). Let d_1, d_2 be relatively prime and assume that $M\otimes R_{d_i}$ is a free $U_*\otimes R_{d_i}$-module on even dimensional generators. Then M is a free U_*-module on even dimensional generators.*

REFERENCES

1. P. E. Conner, *A bordism theory for actions of an abelian group*, Bull. Amer. Math. Soc. **69** (1963), 244–247. MR **26** #4353.

2. G. Hamrick and E. Ossa, *Unitary bordism of monogenic groups and isometries*, Proc. Conference on Transformation Groups, Amherst, 1971 (to appear).

3. P. S. Landweber, *Equivariant bordism and cyclic groups*, Proc. Amer. Math. Soc. **31** (1972), 564–570.

4. R. E. Stong, *Complex and oriented equivariant bordism*, Topology of Manifolds, J. C. Cantrell and C. H. Edwards, Jr. (Editors), Markham, Chicago, Ill., 1970, pp. 291–316.

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