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ON PERFECT SUBFIELDS OVER WHICH A FIELD IS SEPARABLY GENERATED

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ABSTRACT. In this paper, we determine the perfect subfields over which a field is separably generated.

Let a field F of characteristic $p\neq 0$ be an extension of a field E. A separating transcendence basis for the extension F/E is a transcendence basis B of F/E such that F is a separable algebraic extension of $E(B)$. If F/E has a separating transcendence basis, we say that F is separably generated over E.

A field K of characteristic $p\neq 0$ is perfect if $K^p=K$, that is, if each element of K has a unique pth root in K. It is clear that any field F contains a maximal perfect subfield which we denote by F^* and that $F^*=\bigcap_{n=1}^{\infty} F^{p^n}$.

We prove in Theorem 3 that if F is separably generated over a subfield E, then F^* is an algebraic extension of E^*.

As a consequence (Corollary 10), we show that if a field F is separably generated over at least one perfect subfield, then F is separably generated over a perfect subfield K if and only if F^* is algebraic over K.

LEMMA 1. *If F is a pure transcendental extension of E, then $F^*=E^*$.*

PROOF. By hypothesis, $F=E(B)$ where B is an algebraically independent set of elements over E. If y is a nonzero element of F^*, then there exist elements w_n in F such that $y=w_n^{p^n}$ for $n=1, 2, \cdots$. Since the polynomial ring $E[B]$ is a unique factorization domain, we may write $y=f/g$ with f and g relatively prime in $E[B]$. Similarly, for each n, $w_n=f_n/g_n$ with f_n and g_n relatively prime in $E[B]$. Thus $fg_n^{p^n}=gf_n^{p^n}$ implies that f and $f_n^{p^n}$ are associates for each n, that is, $f=a_nf_n^{p^n}$ where a_n is in E. As a consequence of the unique factorization in $E[B]$, it follows that $f\in E$ and $f_n\in E$, for all n. Similarly $g\in E$ and $g_n\in E$, for all n. Thus $y\in E^*=\bigcap_{n=1}^{\infty} E^{p^n}$, proving that $F^*=E^*$.

The following lemma appears as an exercise in [1, Example 16, p. 136]. For the sake of completeness, we provide a proof.

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LEMMA 2. *Let $f(x)$ be a monic irreducible polynomial in $E[x]$, where E is a field of characteristic $p \neq 0$. Then in $E[x]$, the polynomial $f(x^p)$ is irreducible or is the pth power of an irreducible polynomial according to whether or not there exists a coefficient of $f(x)$ which does not belong to E^p.*

PROOF. Let F be an extension field of E which contains a root w of $g(x)=f(x^p)$. If $u=w^p$, then $f(u)=g(w)=0$. Thus the degree of $E(u)$ over E is equal to m, the degree of $f(x)$, since $f(x)$ is irreducible in $E[x]$. Suppose that $g(x)$ is reducible in $E[x]$. Then $E(w)=E(u)$, for w not in $E(u)$ implies that the degree of $E(w)$ over $E(u)$ is p [3, Theorem 7, p. 66] and so the degree of $E(w)$ over E is pm, contradicting the reducibility of $g(x)$. Thus if $h(x)=x^m+b_1x^{m-1}+\cdots+b_m$ is the minimum polynomial of w over E, then $h(w)=0$ and $(h(w))^p=0$, that is, $u^m+b_1^p u^{m-1}+\cdots+b_m^p=0$. Therefore, $f(x)=x^m+b_1^p x^{m-1}+\cdots+b_m^p$ and $f(x^p)=(h(x))^p$, completing the proof.

THEOREM 3. *If a field F is separably generated over a subfield E, then F^* is an algebraic extension of E^*.*

PROOF. If B is a separating transcendence basis of F over E, then F is a separable algebraic extension of $F_1=E(B)$. Let u be any element of F^* and let $f(x)=x^m+a_1x^{m-1}+\cdots+a_m$ be the minimum polynomial of u over F_1. There exists an element w in F^* such that $w^p=u$ and since F is a separable algebraic extension of $F_1(u)$, it follows that w is in $F_1(u)$. If $g(x)=f(x^p)$, then $g(w)=0$ and hence $g(x)$ is reducible in $F_1[x]$. By Lemma 2, $g(x)=(h(x))^p$ where $h(x)=x^m+b_1x^{m-1}+\cdots+b_m$ is irreducible in $F_1[x]$. Thus $h(w)=0$ and $b_i^p=a_i$, for $i=1,\cdots,m$. By repeating the argument for w, we obtain an element w_1 in F^* with $w_1^p=w$ and an irreducible polynomial $h_1(x)=x^m+c_1x^{m-1}+\cdots+c_m$ in $F_1[x]$ such that $h_1(w_1)=0$ and $c_i^p=b_i$, for $i=1,\cdots,m$. Continuing inductively, we have for each i, a sequence of elements $d(i,n)$ in F_1 such that $d_{(i,n)}^{p^n}=a_i$, that is, $a_i \in F_1^*=\bigcup_{n=1}^{\infty} F_1^{p^n}$, for $i=1,\cdots,m$. However Lemma 1 shows that $F_1^*=E^*$, and so $f(x)\in E^*[x]$. Hence u is algebraic over E^* which completes the proof.

EXAMPLE. The following example shows that the equality $F^*=E^*$ does not always hold given only the hypothesis of Theorem 3. If E is the finite field with p elements and F is a simple transcendental extension of the field with p^2 elements, then by [2, Theorem 5], F is separably generated over E, although $F^*\neq E^*$.

We shall also require the following result.

THEOREM 4. *Let K be an extension field of E and F an extension of K. The following statements are equivalent.*

(i) *F is separably generated over E and K is an algebraic extension of E.*

(ii) *F is separably generated over K and K is a separable algebraic extension of E.*

PROOF. Assume that (i) holds and let B be a separating transcendence basis of F over E. We show that B is also a separating transcendence basis of F over K. As F is a separable algebraic extension of $E(B)$ and thus a separable algebraic extension of $K(B)$, it remains to prove that B is a transcendence basis of F over K. However, B contains a transcendence basis B' of F over K and since K is assumed to be algebraic over E, we see that $K(B')$ is algebraic over $E(B')$ and so $K(B)$ is algebraic over $E(B')$. This implies that $E(B)$ is an algebraic extension of $E(B')$ and hence $B=B'$. Moreover, K is a separable algebraic extension of E since $K(B)$ is a separable algebraic extension of $E(B)$.

Conversely, if (ii) holds, then K is clearly separably generated over E and transitivity of separable generation implies that F is separably generated over E.

For a subfield E of a field F, let E_0 denote the subfield of F of elements separable algebraic over E and $\bar{E}$ the algebraic closure of E in F.

COROLLARY 5. *F is separably generated over E if and only if F is separably generated over E_0. If either (and hence both) of these conditions holds, then $E_0=\bar{E}$.*

PROOF. If F is separably generated over E, then Theorem 4 shows that $E_0=\bar{E}$ and F is separably generated over E_0. Conversely if F is separably generated over E_0, then F is separably generated over E.

We remark that the example shows that [**2**, Theorem 19] is incorrect as stated. As Professor Mac Lane has noted in a personal communication, [**2**, Theorem 19] becomes valid with the additional hypothesis that E is algebraically closed in F. We now recover this result from the next corollary.

COROLLARY 6. *If a field F has a separating transcendence basis over a subfield E, then E_0 contains F^*.*

PROOF. By Theorem 3, F^* is a separable algebraic extension of E^* and thus E_0 contains F^*.

COROLLARY 7 (MAC LANE). *If a field F has a separating transcendence basis over a subfield E and E is algebraically closed in F, then E contains the maximal perfect subfield of F.*

COROLLARY 8. *If F/K and K/E are field extensions, then the following are equivalent.*

(1) *F is separably generated over E and K is perfect.*

(2) *E is perfect, K is an algebraic extension of E and F is separably generated over K.*

PROOF. If (1) holds, then Theorem 3 shows that F^* is algebraic over E^*. Since K is perfect, F^* is an extension of K. Thus by Theorem 4, E is perfect since K is a separable algebraic extension of E and F is separably generated over K.

Conversely if (2) holds, then K is a separable algebraic extension of E and K is perfect. Applying Theorem 4, we find that F is separably generated over E, to complete the proof.

For example, if E is a perfect field and $K=E(x, x^{p^{-1}}, x^{p^{-2}}, \cdots)$ where x is transcendental over E, then no extension field of K is separably generated over E.

COROLLARY 9. *Let F be an extension field of E and suppose that F is separably generated over E. If E is separably generated over E^*, then F is separably generated over F^*. Conversely, if F is separably generated over F^* and E has finite transcendence degree over E^*, then E is separably generated over E^*.*

PROOF. If E is separably generated over E^*, then by transitivity, F is separably generated over E^*. Hence F is separably generated over F^* by Corollary 8.

For the converse, note F^* is algebraic over E^* by Theorem 3. If F is separably generated over F^*, then Corollary 8 implies that F is separably generated over E^*. Further, if tr. d. E/E^* is finite, then E is separably generated over E^* by the corollary in [**2**, p. 386].

COROLLARY 10. *Let F be a field and suppose that the set Ω of perfect subfields over which F is separably generated is not empty. Then F^* belongs to Ω. Moreover, a perfect subfield E belongs to Ω if and only if F^* is an algebraic extension of E.*

PROOF. It is evident from Corollary 9 that F^* belongs to Ω and hence Corollary 8 establishes the assertion.

For a given field F, the set Ω may be empty. Indeed Mac Lane [**2**, §10] has shown the existence of a field M which does not have a separating transcendence basis over its maximal perfect subfield M^*, although tr. d. $M/M^*=2$. However, if a field F has transcendence degree 1 or is finitely generated over F^*, then F is always separably generated over F^* [**2**, Theorems 1 and 4].

We ask whether there exists an imperfect field which is separably generated only over itself. Such a field satisfies this property if and only if it is separable algebraic only over itself, since any pure transcendental extension is a separable algebraic extension of a proper subextension.

We close by stating the following consequence of Corollary 8.

COROLLARY 11. *Let F be a field which is separably generated over its prime subfield P. If K is any perfect subfield of F, then K is an algebraic extension of P and F is separably generated over K.*

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