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A LOWER BOUND FOR THE PERMANENT OF A (0, 1)-MATRIX[1]

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ABSTRACT. Let $A=(a_{ij})$ be an n-square fully indecomposable (0, 1)-matrix. It is shown that if each row sum of A is at least k then per $A \geqq \sum_{i,j=1}^{n} a_{ij}-2n+2+\sum_{m=1}^{k-1}(m!-1)$. This improves an inequality obtained by H. Minc.

Lower bounds for the permanent of a (0, 1)-matrix are of considerable combinatorial interest. A well-known theorem of M. Hall **[1]** states that if A is an n-square (0, 1)-matrix with positive permanent and at least k positive entries in each row, then per $A \geqq k!$. Recently H. Minc **[3]** proved that if A is an n-square fully indecomposable (0, 1)-matrix then per $A \geqq \sigma(A)-2n+2$, where $\sigma(A)$ is the sum of all entries of A. In this note we show that Hall's inequality can be used to improve Minc's inequality.

Let $A=(a_{ij})$ be an n-square matrix. Let A^{km} be the n-square matrix obtained from A by replacing a_{km} by 0, let $r(A)$ denote the minimal row sum of A, and let A_{km} be the $(n-1)$-square submatrix of A that remains after row k and column m are removed. If A contains an $s\times(n-s)$ zero submatrix, for some $1\leqq s\leqq n-1$, then A is *partly decomposable*; otherwise, A is *fully indecomposable*. If A is fully indecomposable, while A^{km} is partly decomposable whenever $a_{km}\neq 0$, then A is *nearly decomposable*.

THEOREM. *If A is an n-square fully indecomposable* (0, 1)*-matrix with $r(A)\geqq k$, then*

$$\text{per } A \geqq \sigma(A) - 2n + 2 + \sum_{m=1}^{k-1} (m! - 1). \tag{1}$$

PROOF. The proof is by induction on k. If $k=1$ or 2, then this statement follows from Minc's inequality. Suppose that it holds for all $t<k$, where $k\geqq 3$, and let $A=(a_{ij})$ be an n-square fully indecomposable (0, 1)-matrix with $r(A)\geqq k$. Since $r(A)\geqq 3$, it follows from Hartfiel's form for

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nearly decomposable matrices [**2**] that A is not nearly decomposable. Hence, there exist $i, j \in \{1, \cdots, n\}$ such that $a_{ij}=1$ and A^{ij} is fully indecomposable. Since $r(A^{ij}) \geqq k-1$ and $\sigma(A^{ij})=\sigma(A)-1$, the inductive assumption implies that

$$\text{(2)} \qquad \operatorname{per} A^{ij} \geqq \sigma(A) - 1 - 2n + 2 + \sum_{m=1}^{k-2} (m! - 1).$$

Since A is fully indecomposable, per $A_{ij}>0$. Hence, since $r(A_{ij}) \geqq k-1$, Hall's inequality implies that

$$\text{(3)} \qquad \operatorname{per} A_{ij} \geqq (k-1)!.$$

Since $a_{ij}=1$, per $A=$per $A^{ij}+$per A_{ij}. Combining this with (2) and (3), we have (1).

Let Λ_n^k be the set of all n-square (0, 1)-matrices with precisely k positive entries in each row and each column. Minc [**3**] showed that if $A \in \Lambda_n^k$ then per $A \geqq n(k-2)+2$. Hartfiel [**2**] discovered that if $A \in \Lambda_n^3$ then per $A \geqq n+3$. Using our theorem it is easy to prove the following.

COROLLARY. *If* $A \in \Lambda_n^k$ *then*

$$\operatorname{per} A \geqq n(k-2) + 2 + \sum_{m=1}^{k-1} (m! - 1).$$

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