

PROJECTING $C(S)$ ONTO $C_0(S)$

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ABSTRACT. If a locally compact Hausdorff space S has a denumerable discrete closed subspace N for which there exists a simultaneous extension E from $C(N)$ into $C(S)$ satisfying $E(C_0(N)) \subset C_0(S)$, then $C_0(S)$ is uncomplemented in $C(S)$. This holds whenever (i) S is not pseudocompact, or (ii) S is not countably compact and is a subspace of a basically disconnected space.

Introduction. $C(S)$ denotes the Banach space of bounded continuous real or complex valued functions on S with supremum norm and $C_0(S)$ its subspace of functions vanishing at infinity.

In [C], J. B. Conway proved that S is pseudocompact (i.e. all continuous scalar valued functions on S are bounded) whenever $C_0(S)$ is complemented in $C(S)$ (i.e. $C(S)$ can be projected onto $C_0(S)$ in the sense of [DS, p. 480]). This generalizes the well-known theorem of R. S. Phillips [P, p. 539] that (c_0) is uncomplemented in (m) .

In §1 we prove some sufficient conditions for $C_0(S)$ to be uncomplemented in $C(S)$ and show how Conway's theorem can be derived from our Theorem 1.3. Our results are formulated in terms of the existence of certain simultaneous extensions. As is well known, the existence of such an operator from $C(\beta S - S)$ into $C(\beta S)$ (βS is the Stone-Čech compactification of S) is equivalent to $C_0(S)$ being complemented in $C(S)$ (cf. [D, Theorem 1]).

In §2, we apply these results to certain totally disconnected spaces. We show in Theorem 2.2 that if S is a subspace of some basically disconnected space, then S is countably compact whenever $C_0(S)$ is complemented in $C(S)$. We also prove a general result, Theorem 2.7, concerning simultaneous extensions in extremally disconnected spaces.

Background. Let A be a subset of S .

$C(S|_A)$ denotes the space of all functions in $C(S)$ that are zero on A .

A is *strongly* (resp. *normally*) *embedded* in S if every continuous (resp. bounded continuous) scalar valued function on A has a continuous extension to S . We identify $\text{cl}_{\beta S} A$ with βA whenever A is normally embedded in S [GJ, p. 89].

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$\Sigma(A, S)$ denotes the set of all bounded linear transformations E from $C(A)$ into $C(S)$ such that the restriction of Ef to A equals f for all $f \in C(A)$. Such an operator is a *simultaneous extension*. Even for compact A and S , $\Sigma(A, S)$ may be empty. Indeed, the above-mentioned theorem of Phillips may be restated as $\Sigma(\beta N - N, \beta N) = \emptyset$ where N is the denumerable discrete space. If, however, A is a *retract* of S (the image of S under a continuous mapping r which is the identity on A) then $Ef = f \circ r$, $f \in C(A)$, defines an E in $\Sigma(A, S)$.

We do not distinguish between the spaces $C(S)$ and $C(\beta S)$ or the spaces $C_0(S)$ and $C(\beta S \parallel \beta S - S)$. We also identify, when convenient, naturally corresponding elements of $\Sigma(A, S)$, $\Sigma(A, \beta S)$, and, when A is normally embedded in S , $\Sigma(\beta A, \beta S)$.

1. Throughout this section, S is a locally compact Hausdorff space and N is a denumerable, discrete (in the relative topology) subspace of S . Such an N always exists when S is infinite and the existence of an N closed in S is equivalent to S not being countably compact.

THEOREM 1.1. *If N is closed and normally embedded in S and $\Sigma(\beta N - N, \beta S - S) \neq \emptyset$ then $\Sigma(\beta S - S, \beta S) = \emptyset$.*

PROOF. If $E \in \Sigma(\beta N - N, \beta S - S)$ and $F \in \Sigma(\beta S - S, \beta S)$ then $RFE \in \Sigma(\beta N - N, \beta N)$, where R is the restriction operator from $C(\beta S)$ onto $C(\beta N)$. This contradicts Phillips' theorem.

COROLLARY 1.2. *If N is closed and normally embedded in S and $\beta N - N$ is a retract of $\beta S - S$ then $\Sigma(\beta S - S, \beta S) = \emptyset$.*

Corollary 1.2 is useful in applications since $\beta N - N$ is a retract of $\beta S - S$ whenever the two sets are equal. For example, this yields immediately the result [C, Example 3] that $C_0(S)$ is uncomplemented in $C(S)$ when S is the pseudocompact space $\beta R - (\beta N - N)$ of [GJ, 6P].

THEOREM 1.3. *If N is closed and normally embedded in S and there exists $E \in \Sigma(N, S)$ such that $E(C_0(N)) \subset C_0(S)$, then $\Sigma(\beta S - S, \beta S) = \emptyset$.*

PROOF. By Theorem 1.1, it suffices to show that $\Sigma(\beta N - N, \beta S - S) \neq \emptyset$.

For each $f \in C(\beta N - N)$, let $Ff = REf^*$, where f^* is any continuous extension of f to βN and R is the restriction operator from $C(\beta S)$ onto $C(\beta S - S)$. The condition $E(C(\beta N \parallel \beta N - N)) \subset C(\beta S \parallel \beta S - S)$ insures that F is well defined and it is easily verified that $F \in \Sigma(\beta N - N, \beta S - S)$.

If S is not pseudocompact it contains a strongly embedded N and such an N is necessarily closed in S . Hence Conway's theorem is an immediate consequence of Theorem 1.3 and the following.

LEMMA 1.4. *Whenever N is strongly embedded in S , there exists $E \in \Sigma(N, S)$ such that $E(C_0(N)) \subset C_0(S)$.*

PROOF. Let $N = \{p_n\}$. Since N is strongly embedded we can choose a sequence $\{U_n\}$ of pairwise disjoint open sets, with $p_n \in U_n$, such that $\{\text{cl } U_n\}$ is a locally finite family of compact sets [BCM, Theorem 1]. Choose any $u_n \in C(S)$ such that $u_n(p_n) = 1$ and u_n is zero on $S - U_n$. It is easily verified that the formula

$$Ef = \sum_{n=1}^{\infty} f(p_n)u_n, \quad f \in C(N),$$

defines the required simultaneous extension.

COROLLARY 1.5. *If N is closed in S and there exists a locally compact space Q which is dense and normally embedded in S such that N is strongly embedded in Q , then $\Sigma(\beta S - S, \beta S) = \emptyset$.*

PROOF. By Lemma 1.4, there exists $F \in \Sigma(N, Q)$ such that $F(C_0(N)) \subset C_0(Q)$. Since $\beta Q = \beta S$, F induces an $E \in \Sigma(N, S)$ in a natural way. Since $\beta S - S \subset \beta Q - Q$, it follows that $E(C_0(N)) \subset C_0(S)$.

It remains an open question as to whether the existence of a closed, normally embedded N in S always implies that $\Sigma(\beta S - S, \beta S) = \emptyset$. The following example, suggested by J. R. Isbell, shows that this hypothesis does not insure the existence of a subspace Q satisfying the hypothesis of Corollary 1.5.

EXAMPLE 1.6. Let W be the space of all ordinals less than the first uncountable ordinal Ω . Then the space of all ordinals less than or equal to Ω may be identified with βW [GJ, 5.12]. If M is any denumerable discrete space, let $S = (\beta W \times \beta M) - (\{\Omega\} \times \beta M - M)$, and let $N = \{\Omega\} \times M$. It can be verified in a rather lengthy but straightforward manner that

- (a) S is locally compact and $\beta S = \beta W \times \beta M$;
- (b) N is closed and normally embedded in S ; and
- (c) N is strongly embedded in any subspace Q of $\beta W \times \beta M$ containing N such that $\beta Q = \beta S$.

Note that $\Sigma(\beta S - S, \beta S) = \emptyset$ by Corollary 1.2 since $\beta S - S = \beta N - N$.

2. A *clopen* subset of a topological space is one that is both open and closed. A *zero set* is a set on which some continuous scalar valued function is zero, and a *cozero set* is the complement of a zero set.

A completely regular Hausdorff space is *extremally* (resp. *basically*) *disconnected* if every open (resp. cozero) set has an open closure. For background on these spaces see [GJ]. Clearly, every extremally disconnected space is basically disconnected and every subspace of a basically disconnected space has a base of clopen sets. However, even a compact

subspace of an extremally disconnected space need not be basically disconnected [GJ, 6W].

LEMMA 2.1. *Let S be a locally compact subspace of a basically disconnected space and N a denumerable discrete subspace of S . Then there exists a multiplicative $E \in \Sigma(N, S)$ such that $E(C_0(N)) \subset C_0(S)$.*

PROOF. Since the Stone-Čech compactification of a basically disconnected space is basically disconnected, we may assume that S is contained in some compact, basically disconnected space K . If $N = \{p_n\}$, choose a sequence $\{U_n\}$ of open sets in S such that $p_n \in U_n$ and $\{cl_S U_n\}$ is a family of compact, pairwise disjoint sets. If W_n is any open set in K such that $U_n = S \cap W_n$, we can choose a sequence of pairwise disjoint, clopen subsets of K , $\{V_n\}$, such that $p_n \in V_n \subset W_n$. It follows that $p_n \in S \cap V_n \subset U_n$.

Since $cl_K \bigcup_{n=1}^{\infty} V_n = \beta(\bigcup_{n=1}^{\infty} V_n)$, which is clopen, the formulas

$$\begin{aligned}
 Ff(p) &= f(p_n), & p \in V_n, \\
 Ff(p) &= 0, & p \in K - cl_K \bigcup_{n=1}^{\infty} V_n, \quad f \in C(N),
 \end{aligned}$$

define a unique multiplicative $F \in \Sigma(N, K)$. Let $E = RF$ where R is the restriction to S . Obviously, $E \in \Sigma(N, S)$ and E is multiplicative. It remains to show that $E(C_0(N)) \subset C_0(S)$.

Suppose $f \in C_0(N)$ and $\epsilon > 0$. Choose m such that $|Ff(p_n)| < \epsilon$ for $n \geq m$. It follows that $|Ff(p)| < \epsilon$ for all $p \in \bigcup_{n=m}^{\infty} V_n$ and, by continuity of Ff , we have $|Ff(p)| \leq \epsilon$ for all $p \in cl_K \bigcup_{n=m}^{\infty} V_n$. Now,

$$S \cap cl_K \bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{m-1} (S \cap V_n) \cup \left(S \cap cl_K \bigcup_{n=m}^{\infty} V_n \right).$$

Therefore, $|Ef(p)| \leq \epsilon$ for all $p \in S \cap cl_K \bigcup_{n=m}^{\infty} V_n$, and $Ef(p) = 0$ for $p \in S - cl_K \bigcup_{n=1}^{\infty} V_n$. Since $\bigcup_{n=1}^{m-1} (S \cap V_n) \subset \bigcup_{n=1}^{m-1} cl_S U_n$, and the latter set is compact, it follows that $Ef \in C_0(S)$.

THEOREM 2.2. *If S is a locally compact subspace of a basically disconnected space and $C_0(S)$ is complemented in $C(S)$ then S is countably compact.*

PROOF. If S is not countably compact it contains a denumerable discrete closed subspace N . By Lemma 2.1 there exists $E \in \Sigma(N, S)$ with $E(C_0(N)) \subset C_0(S)$. Hence Theorem 1.3 implies that $\Sigma(\beta S - S, \beta S) = \emptyset$.

COROLLARY 2.3. *Let K be a compact subspace of a basically disconnected space and N a denumerable discrete subspace of K . If $S = K - (cl_K N - N)$ then $C_0(S)$ is uncomplemented in $C(S)$.*

PROOF. S is clearly locally compact, and since N is closed in S , Theorem 2.2 yields the desired conclusion.

As the following example shows, the converse of Conway's theorem is false even for extremally disconnected spaces.

EXAMPLE 2.4. There exists a locally compact, pseudocompact extremally disconnected space S that is not countably compact. Hence $C_0(S)$ is uncomplemented in $C(S)$ by Theorem 2.2.

Let M be any denumerable discrete subspace of $\beta N - N$. Then $\beta M = \text{cl } M$ and $(\beta N - N) - (\beta M - M)$ is open and dense in $\beta N - N$. Let $S = \beta N - (\beta M - M)$. It follows from [FG₂, Theorem 3.1] that S is pseudocompact. Clearly, S is locally compact and $\beta S = \beta N$. Since N is extremally disconnected, S is also. S is not countably compact since M is closed in S .

In the following, $\text{fr } A$ denotes the boundary of A .

LEMMA 2.5. Let A be a closed subset of a compact, extremally disconnected space K . If $S = K - A$, then the following are equivalent.

- (a) $\Sigma(A, K) \neq \emptyset$.
- (b) $\Sigma(\text{fr } A, K) \neq \emptyset$.
- (c) $\Sigma(\beta S - S, \beta S) \neq \emptyset$.

PROOF. We note that S is locally compact and extremally disconnected. Since S is open in K , $\text{cl } S = \beta S$ is clopen in K . Therefore, $\Sigma(\beta S, K) \neq \emptyset$. Also, $\beta S - S = A \cap \beta S$ is clopen in A . Therefore, $\Sigma(\beta S - S, A) \neq \emptyset$. We also note that $\text{fr } A = \text{fr}(K - A) = \text{fr } S = \beta S - S$.

(a) implies (b). (b) follows from (a) and $\Sigma(\beta S - S, A) \neq \emptyset$.

(b) implies (c). $\beta S - S \subset \beta S \subset K$.

(c) implies (a). If $F \in \Sigma(\beta S - S, \beta S)$, $f \in C(A)$, and Rf is the restriction of f to $\beta S - S$, define $Ef = f$ on $A - \beta S$, $Ef = FRf$ on βS . $E \in \Sigma(A, K)$ since βS is clopen in K .

Note that S is countably compact whenever the above conditions are satisfied.

THEOREM 2.6. If A is any closed subset of $\beta N - N$ having nonempty interior then $\Sigma(A, \beta N) = \emptyset$.

PROOF. Since A has nonempty interior there exists an infinite subset M of N such that $\beta M - M \subset A$ (cf. [GJ, 6S4]). Such an M is closed in $\beta N - A$ so the conclusion follows from Lemma 2.5 and the above remark.

A completely regular Hausdorff space is *realcompact* if it is homeomorphic with a closed subset of a product of real lines [GJ, Chapter 8]. A pseudocompact realcompact space is necessarily compact [GJ, 5H2].

THEOREM 2.7. Let A be a closed subset of a compact extremally disconnected space K and let $S = K - A$. If S is realcompact, then the following are equivalent.

- (a) $\Sigma(A, K) \neq \emptyset$.
- (b) A is open in K .
- (c) A is extremally disconnected.
- (d) A is basically disconnected.

PROOF. It is obvious that (b) implies (a) and (c), and (c) implies (d). We shall show that (a) and (d) each imply (b).

If A is not open then S is not compact. Hence $\beta S - S$ is not basically disconnected [FG₁, Remark 3.2]. Since $\beta S - S$ is clopen in A , it follows that A is not basically disconnected. Therefore, (d) implies (b).

Now, S realcompact and not compact implies that S is not pseudo-compact. Therefore, $\Sigma(\beta S - S, \beta S) = \emptyset$ and it follows from Lemma 2.5 that $\Sigma(A, K) = \emptyset$. Hence (a) implies (b).

Note that S is realcompact in Theorem 2.7 whenever A is a zero set of K [GJ, 8.14]. The equivalence of (a) and (b) in his case follows from a result of W. Bade [B, Corollary 3.3].

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