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PURE STATES WITH THE RESTRICTION PROPERTY

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ABSTRACT. Conditions are given which imply that a pure state of a B^*-algebra A restricts to a pure state of some maximal commutative $*$-subalgebra of A.

1. **Introduction.** A pure state ρ of a B^*-algebra A has *the restriction property* if there exists a maximal commutative $*$-subalgebra C of A such that the restriction of ρ to C is a pure state of C (i.e. ρ is a nonzero multiplicative linear functional on C). The work of R. Kadison and I. Singer in [**4**] raises the question of whether or not each pure state of a B^*-algebra has the restriction property. This question was answered by J. Aarnes and R. Kadison for a special class of B^*-algebras A. They prove that when A is separable and has an identity, then each pure state of A has the restriction property [**1**, Theorem 2]. Again in the case when A is separable, C. Akemann in [**2**] removed the requirement that A have an identity and made other improvements in the result of Aarnes and Kadison (including a proof that in this case a pure state ρ of A is the *unique* extension of a pure state of some maximal commutative $*$-subalgebra of A). However, the general question remains open.

In this note we give several new conditions on a pure state ρ of a B^*-algebra which imply that ρ has the restriction property. A is a B^*-algebra throughout. Let $a \to \pi(a)$ be a $*$-representation of A on a Hilbert space $\mathscr{H}$. A positive functional ρ is represented by π if there is $\xi \in \mathscr{H}$, $\|\xi\| = 1$, such that $\rho(a) = (\pi(a)\xi, \xi)$ for all $a \in A$. A pure state of A is always represented by some irreducible $*$-representation of A; see [**3**, pp. 32, 33, 37] for details. Now let ρ be a pure state of A which is represented by an irreducible $*$-representation π of A on a Hilbert space $\mathscr{H}$. We prove that if either $\mathscr{H}$ is separable or $\pi(A)$ contains $\mathscr{F}(\mathscr{H})$, the algebra of bounded operators on $\mathscr{H}$ with finite dimensional range, then ρ has the restriction property. The proofs of these results are indebted to the ideas of Aarnes and Kadison in [**1**].

2. **The results.** Let be $\mathscr{H}$ a Hilbert space. $\mathscr{B}(\mathscr{H})$ is the algebra of bounded operators on $\mathscr{H}$. When $\mathscr{K}$ is a subspace of $\mathscr{H}$ and B is a

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nonempty subset of $\mathscr{B}(\mathscr{H})$, then $B\mathscr{H}$ is the linear span of the vectors $\{T\psi \mid T \in B, \psi \in \mathscr{H}\}$. $[B\mathscr{H}]$ is the closure of $B\mathscr{H}$ in $\mathscr{H}$. When $T \in \mathscr{B}(\mathscr{H})$, then $\mathscr{N}(T)$ is the null space of T and $\mathscr{R}(T)$ is the range of T.

LEMMA. *Let $\mathscr{H}$ be a separable Hilbert space and assume that A is a closed *-subalgebra of $\mathscr{B}(\mathscr{H})$ such that $[A\mathscr{H}]=\mathscr{H}$. Then there exists $T \in A$, $T \geqq 0$, such that $\mathscr{N}(T)=0$.*

PROOF. Since $[A\mathscr{H}]=\mathscr{H}$, then given any $\psi \in \mathscr{H}$, $\psi \neq 0$, there exists $S \in A$ such that $S\psi \neq 0$. Then also $S^*S\psi \neq 0$. Therefore for each $\psi \in \mathscr{H}$, $\psi \neq 0$, we can choose $T_\psi \in A$ such that $T_\psi \geqq 0$ and $T_\psi(\psi) \neq 0$. Let $U_\psi = \{\xi \in \mathscr{H} \mid T_\psi(\xi) \neq 0\}$. The collection $\{U_\psi \mid \psi \in \mathscr{H}, \psi \neq 0\}$ is an open cover for $\mathscr{H}\backslash\{0\}$. $\mathscr{H}\backslash\{0\}$ being a separable metric space is Lindelöf (every open cover has a countable subcover). It follows that there exists a sequence $\{T_n\} \subset A$ such that $T_n \geqq 0$ and $\bigcap_{n=1}^{+\infty} \mathscr{N}(T_n)=0$. Let $a_n=(2^n\|T_n\|)^{-1}$, and set $T=\sum_{n=1}^{+\infty} a_n T_n$. If $T\psi=0$, then $\sum_{n=1}^{+\infty} a_n(T_n\psi, \psi)=0$. Therefore $(T_n\psi, \psi)=0$ for each n. But then $T_n^{1/2}\psi=0$, which implies $T_n\psi=0$ for each n. Therefore $\psi=0$.

When D is a nonempty subset of A, we let

$$\mathscr{C}(D) = \{a \in A \mid ad = da \text{ for all } d \in D\}.$$

If D is selfadjoint, then $\mathscr{C}(D)$ is a closed *-subalgebra of A.

THEOREM 1. *Let $a \rightarrow \pi(a)$ be an irreducible *-representation of A on a separable Hilbert space $\mathscr{H}$. If ρ is a positive functional represented by π, then ρ has the restriction property.*

PROOF. There exists $\xi \in \mathscr{H}$, $\|\xi\|=1$, such that $\rho(a)=(\pi(a)\xi, \xi)$ for all $a \in A$. Let $K=\{a \in A \mid \rho(a^*a)=0\}=\{a \in A \mid \pi(a)\xi=0\}$. Set $A_0=K \cap K^*$ and $\mathscr{H}_0=\{\xi\}^\perp$. Given $a \in A_0$ and $\psi \in \mathscr{H}$, we have $(\pi(a)\psi, \xi)=(\psi, \pi(a^*)\xi)=0$. Therefore $\pi(A_0)\mathscr{H} \subset \mathscr{H}_0$. Let E be the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}_0$. Then

$$E\pi(a) = \pi(a) \quad \text{for all } a \in A_0. \tag{1}$$

By [**3**, Corollaire (2.8.4)], $\pi(A)$ acts strictly irreducibly on $\mathscr{H}$. Therefore there exists $v \in A$ such that $\pi(v)\xi=\xi$. Since $\pi(v+v^*-v^*v)\xi=\xi$, we may assume that $v=v^*$. Set $u=2v-v^2$. Then $I-\pi(u)=(I-\pi(v))^2 \geqq 0$ where I is the identity operator on $\mathscr{H}$. If $\psi \in \mathscr{H}$, $((I-\pi(u))\psi, \xi)=(\psi, (I-\pi(v))^2\xi)=0$. Therefore

$$E(I - \pi(u)) = I - \pi(u). \tag{2}$$

Given $\psi \in \mathscr{H}_0$, the transitivity theorem [**3**, Théorème (2.8.3)] implies that there exists $a \in A$ such that $a=a^*$, $\pi(a)\xi=0$, and $\pi(a)\psi=\psi$. Then $a \in A_0$, and this proves that $\pi(A_0)\mathscr{H}_0=\mathscr{H}_0$. By the Lemma there exists $w \in A_0$, $w \geqq 0$, such that $\mathcal{N}(\pi(w))\cap\mathcal{H}_0=0$. Set $S=I-\pi(u)+\pi(w)$. Since $I-\pi(u)\geqq 0$, then $\mathcal{N}(S)\cap\mathcal{H}_0=0$. Let $y=u-w$, and choose C_0 a maximal commutative *-subalgebra of $\mathscr{C}(y)\cap A_0$. Let C be the closed commutative *-subalgebra of A generated by y and C_0. We prove that C is a maximal commutative *-subalgebra of A. Assume that $b=b^*$ and $b\in\mathscr{C}(C)$. Let $b_0=b-\rho(b)y$. Then $b_0=b_0^*$ and $b_0\in\mathscr{C}(C)$. Using (1) and (2) we have $ES=E(I-\pi(u)+\pi(w))=I-\pi(u)+\pi(w)=S$. Also $\pi(b_0)S=S\pi(b_0)$. Then

$$(E\pi(b_0)-\pi(b_0)E)S=S\pi(b_0)-\pi(b_0)S=0.$$

Since $\mathcal{N}(S)\cap\mathcal{H}_0=0$ and $S=S^*$, then $(\mathscr{R}(S))^-=\mathcal{H}_0=\mathscr{R}(E)$. Therefore $(E\pi(b_0)-\pi(b_0)E)E=0$. It follows that $\pi(b_0)E=E\pi(b_0)$. Then there exists a scalar λ such that $\pi(b_0)\xi=\lambda\xi$. Note that $\rho(y)=\rho(u-w)=\rho(u)=\rho(2v-v^2)=(\pi(2v-v^2)\xi,\ \xi)=1$. Therefore $\lambda=(\pi(b_0)\xi,\ \xi)=\rho(b_0)=\rho(b-\rho(b)y)=0$. Then $b_0\in\mathscr{C}(y)\cap A_0$, and it follows that $b_0\in C_0$. But then $b\in C$. This proves that C is a maximal commutative *-subalgebra of A.

ρ is nonzero on C since $\rho(y)=1$. It remains to be shown that ρ is multiplicative on C. Given $a\in C_0$, then $ya\in\mathscr{C}(y)$. Also $\pi(ya)\xi=\pi(y)\pi(a)\xi=0$ and similarly $\pi(a^*y)\xi=0$. Therefore ya and $(ya)^*$ are in $\mathscr{C}(y)\cap A_0$. Thus $ya\in C_0$. Furthermore $\pi(y)\xi=(\pi(u)-\pi(w))\xi=\pi(u)\xi=\xi$. Thus $\pi(y^n-y)\xi=0$ for any positive integer n. Then $y^n-y\in\mathscr{C}(y)\cap A_0$, and therefore $y^n-y\in C_0$ for each positive integer n. It follows that every element of C has the form $\lambda y+a$ for some scalar λ and some $a\in C_0$. Then given λ, μ scalars and a, $b\in C_0$,

$$\rho((\lambda y+a)(\mu y+b))=\lambda\mu=\rho(\lambda y+a)\rho(\mu y+b).$$

This completes the proof of the theorem.

In the case where A has an identity, the proof of Theorem 1 can be considerably simplified.

THEOREM 2. *Let $a\rightarrow\pi(a)$ be a *-representation of A on a Hilbert space $\mathcal{H}$ with the property that $\mathscr{F}(\mathcal{H})\subset\pi(A)$. If ρ is a positive functional represented by π, then ρ has the restriction property.*

PROOF. Assume that $\rho(a)=(\pi(a)\xi,\ \xi)$ for all $a\in A$, where $\xi\in\mathcal{H}$, $\|\xi\|=1$. Let $K=\{a\in A\,|\,\pi(a)\xi=0\}$, and set $A_0=K\cap K^*$. Let E be the orthogonal projection with one dimensional range containing ξ. By hypothesis there exists $e\in A$, $e=e^*$, such that $\pi(e)=E$. Choose C_0 a maximal commutative *-subalgebra of $\mathscr{C}(e)\cap A_0$. Let C be the closed commutative *-subalgebra of A generated by e and C_0. Assume that $b=b^*\in\mathscr{C}(C)$. Set $b_0=b-\rho(b)e$. Note that $\rho(e)=(E\xi,\ \xi)=1$, so that $\rho(b_0)=0$. Then $\pi(b_0)E=E\pi(b_0)$. Therefore there exists a scalar λ such that $\pi(b_0)\xi=\lambda\xi$. Then $\lambda=(\pi(b_0)\xi,\ \xi)=\rho(b_0)=0$. It follows that $b_0\in\mathscr{C}(e)\cap A_0$, so that by the definition of C_0, $b_0\in C_0$. Then $b\in C$. This proves that C is a maximal commutative *-subalgebra of A. The proof that ρ is a nonzero multiplicative functional on C proceeds as in the last paragraph of the proof of Theorem 1 with e in place of y.

When A is a GCR algebra (postliminaire) and $a \rightarrow \pi(a)$ is an irreducible *-representation of A on a Hilbert space $\mathscr{H}$, then it is well known that $\mathscr{F}(\mathscr{H}) \subset \pi(A)$; see [**3**, Théorème (4.3.7)]. Therefore we have as a corollary of Theorem 2:

COROLLARY. *A pure state of a GCR algebra A has the restriction property.*

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