PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 2, June 1972

ANNIHILATOR IDEALS IN THE COHOMOLOGY OF BANACH ALGEBRAS

A. M. SINCLAIR

ABSTRACT. If A is a C^*-algebra, if X is a Banach A-module, and if J is the annihilator of X in A, then the cohomology space $\mathscr{H}^n(A, X^*)$ is isomorphic to $\mathscr{H}^n(A/J, X^*)$ for each positive integer n.

B. E. Johnson [5, Proposition 1.8] has shown that if A is a Banach algebra with a bounded approximate identity and if X is a Banach A-module, then $X_1 = \{axb : x \in X;\ a, b \in A\}$ is a closed neo-unital submodule of X and $\mathscr{H}^n(A, X^*)$ is isomorphic to $\mathscr{H}^n(A, X_1^*)$. In particular this result shows that in calculating cohomologies of dual Banach modules for C^*-algebras attention may be restricted to neo-unital modules. Since each closed (two-sided) ideal in a C^*-algebra has a bounded approximate identity [2, Propositions 1.8.2 and 1.7.2] our result shows that for C^*-algebras and dual Banach modules attention may be restricted to faithful modules. If A is a Banach algebra, if X is a Banach A-module, and if J is a closed ideal in A annihilating X, then there is a natural homomorphism Q, which is defined in Theorem 1, from $\mathscr{H}^n(A/J, X^*)$ into $\mathscr{H}^n(A, X^*)$. Under the additional assumption that J has a bounded approximate identity, this homomorphism Q is an isomorphism (Theorem 1). In Remark 4 we give an elementary example to show that an additional assumption on J is necessary if the conclusion of Theorem 1 is to hold.

I am grateful to B. E. Johnson for a preprint of [5], and I acknowledge a C.S.I.R. travel grant.

If A is a Banach algebra, if X is a Banach A-module, and if n is a positive integer, we let $\mathscr{L}^n(A, X^*)$ denote the Banach space of continuous n-linear mappings from A into X^*, the dual of X (our notation and definitions are from [5]). Recall that $\mathscr{L}^n(A, X^*)$ is the dual space of a Banach space $A \hat{\otimes} A \hat{\otimes} \cdots \hat{\otimes} A \hat{\otimes} X$ (see [5]). We also give X^* the dual A-module structure from the Banach A-module X by defining af and fa for a in A and f in X^* by

$$\langle af, x\rangle = \langle f, xa\rangle \quad \text{and} \quad \langle fa, x\rangle = \langle f, ax\rangle$$

Received by the editors August 9, 1971.
AMS 1970 *subject classifications*. Primary 46H25, 18H15.
Key words and phrases. Banach algebra, Banach module, cohomology, annihilator ideal, approximate identity, C^*-algebra.

for all x in X. The mapping δ^n from $\mathscr{L}^{n-1}(A, X^*)$ to $\mathscr{L}^n(A, X^*)$ is defined by

$$(\delta^n T)(a_1, \cdots, a_n) = a_1 T(a_2, \cdots, a_n) \tag{1}$$
$$+ \sum_{j=1}^{n-1} (-1)^j T(a_1, \cdots, a_j a_{j+1}, \cdots, a_n)$$
$$+ (-1)^n T(a_1, \cdots, a_{n-1}) a_n$$

for T in $\mathscr{L}^{n-1}(A, X^*)$, which we take to be X^* when $n-1=0$, and for $a_1, \cdots, a_n$ in A. Then $\delta^{n+1}\delta^n=0$, and we let

$$\mathscr{H}^n(A, X^*) = \operatorname{Ker} \delta^{n+1}/\operatorname{Im} \delta^n.$$

We use the same δ^n for all algebras and modules. We say that an ideal J in a Banach algebra A annihilates a Banach A-module X if $ax=xa=0$ for all a in J and x in X. If the closed ideal J annihilates X, we regard X as a Banach A/J-module by defining $(a+J)x=ax$ and $x(a+J)=xa$ for all a in A and x in X.

THEOREM 1. *Let A be a Banach algebra, let X be a Banach A-module, and let J be a closed ideal in A annihilating X. If J has a bounded approximate identity, then $\mathscr{H}^n(A/J, X^*)$ is isomorphic to $\mathscr{H}^n(A, X^*)$ under the mapping*

$$Q: T + \operatorname{Im} \delta^n \to \theta T + \operatorname{Im} \delta^n$$

where $(\theta T)(a_1, \cdots, a_n) = T(a_1+J, \cdots, a_n+J)$ for T in $\mathscr{L}^n(A/J, X^*)$ and $a_1, \cdots, a_n$ in A.

We require a lemma before proving Theorem 1. Under the hypotheses of Theorem 1 we shall regard $\mathscr{L}^1(A/J, X^*)$ as a Banach A-module by defining aT and Ta, for all a in A and T in $\mathscr{L}^1(A/J, X^*)$, by

$$(aT)(b + J) = aT(b + J) \quad \text{and} \tag{2}$$
$$(Ta)(b + J) = T(ab + J) - T(a + J)b$$

for all b in A. Compare the following lemma with [5, 1.a].

LEMMA 2. *Let A be a Banach algebra, let X be a Banach A-module, and let J be a closed ideal in A annihilating X. Let n be an integer greater than 1. If J has a bounded approximate identity, then $\mathscr{H}^{n-1}(A, \mathscr{L}^1(A/J, X^*))$ is isomorphic to $\mathscr{H}^n(A, X^*)$ under the mapping $\chi: T+\operatorname{Im} \delta^{n-1} \to \psi_n T + \operatorname{Im} \delta^n$ where*

$$(\psi_n T)(a_1, \cdots, a_n) = T(a_1, \cdots, a_{n-1})(a_n + J) \tag{3}$$

for all T in $\mathscr{L}^{n-1}(A, \mathscr{L}^1(A/J, X^*))$ and all $a_1, \cdots, a_n$ in A.

PROOF. A routine calculation using equations (1), (2), and (3) shows that

$$\psi_{n+1}\delta^n = \delta^{n+1}\psi_n \tag{4}$$

for n a positive integer. Applying equation (4) with n replaced by $n-1$ we observe that the mapping χ is well defined. Using (4) as it stands we observe that χ maps $\mathscr{H}^{n-1}(A, \mathscr{L}^1(A/J, X^*))$ into $\mathscr{H}^n(A, X^*)$. We now use the bounded approximate identity in J to show that χ is an isomorphism.

We shall show that there is an R in $\mathscr{L}^{n-1}(A, X^*)$ with

$$(T - \delta^n R)(a_1, \cdots, a_n) = 0 \tag{5}$$

if a_n is in J. If we have found such an R, then equation (5) implies that $T-\delta^n R$ is in the image of ψ_n, and so there is a P in $\mathscr{L}^{n-1}(A, \mathscr{L}^1(A/J, X^*))$ with $\psi_n P = T - \delta^n R$. From equation (4) we obtain

$$\psi_{n+1}\delta^n P = \delta^{n+1}\psi_n P = \delta^{n+1}(T - \delta^n R) = 0,$$

and thus $\delta^n P = 0$ because ψ_{n+1} is a monomorphism. Therefore χ is an epimorphism.

Let T be in $\mathscr{L}^n(A, X^*)$ with $\delta^{n+1}T=0$, and let $\{e_\alpha\}$ be a bounded approximate identity in J. Now $\mathscr{L}^n(A, X^*)$ may be regarded as the dual space of $A\hat{\otimes}\cdots\hat{\otimes}A\hat{\otimes}X$, where there are n-copies of A, and under this identification the weak-*-topology on $\mathscr{L}^n(A, X^*)$ is generated by the seminorms $S \rightarrow \langle S(a_1, \cdots, a_n), x\rangle$ where $\langle\cdot, \cdot\rangle$ denotes the pairing of X and X^*, and $a_1, \cdots, a_n$ are in A and x is in X (see [**5**, §1]). Now $\{T(\cdot, \cdots, e_\alpha)\}$ is a bounded net in $\mathscr{L}^{n-1}(A, X^*)$ and hence has a subnet convergent in the weak-*-topology of $\mathscr{L}^{n-1}(A, X^*)$. For convenience, we take $\{T(\cdot, \cdots, e_\alpha)\}$ itself to be convergent in the weak-*-topology to an element $(-1)^{n+1}R$ in $\mathscr{L}^{n-1}(A, X^*)$. All limits in this proof are over the directed set corresponding to the net $\{e_\alpha\}$ and are in the weak-*-topology in X^*.

Using $\delta^{n+1}T(a_1, \cdots, a_n, e_\alpha)=0$, equation (1), and the definition of R, we obtain

$$\begin{aligned}
&\delta^n R(a_1, \cdots, a_n)\\
&= (-1)^{n+1}\lim\Big\{a_1 T(a_2, \cdots, a_n, e_\alpha) + \sum_{j=1}^{n-1}(-1)^j T(a_1, \cdots, a_j a_{j+1}, \cdots, a_n, e_\alpha)\\
&\qquad\qquad + (-1)^n T(a_1, \cdots, a_{n-1}, e_\alpha)a_n\Big\}\\
&= (-1)^n \lim\{(-1)^n T(a_1, \cdots, a_{n-1}, a_n e_\alpha) + (-1)^{n+1}T(a_1, \cdots, a_n)e_\alpha\\
&\qquad\qquad + (-1)^{n+1}T(a_1, \cdots, a_{n-1}, e_\alpha)a_n\}\\
&= T(a_1, \cdots, a_n)
\end{aligned}$$

provided a_n is in J. This proves equation (5).

We shall now prove that χ is one-to-one. We let T be in

$$\mathscr{L}^{n-1}(A, \mathscr{L}^1(A/J, X^*))$$

with $\psi_n T=\delta^n S$ where S is some element of $\mathscr{L}^{n-1}(A, X^*)$. We shall obtain an R in $\mathscr{L}^{n-2}(A, X^*)$ such that

$$(S-\delta^{n-1}R)(a_1, \cdots, a_{n-1}) = 0 \tag{6}$$

if a_{n-1} is in J. Having found such an R there is a P in $\mathscr{L}^{n-2}(A, \mathscr{L}^1(A/J, X^*))$ such that $\psi_{n-1}P=S-\delta^{n-1}R$. From this and equation (4) we obtain $\psi_n T=\delta^n S=\delta^n\psi_{n-1}P+\delta^n\delta^{n-1}R=\psi_n\delta^{n-1}P$. Because ψ_n is a monomorphism, T is equal to $\delta^{n-1}P$.

We let T be in $\mathscr{L}^{n-1}(A, \mathscr{L}^1(A/J, X^*))$ with $\psi_n T=\delta^n S$ where S is some element of $\mathscr{L}^{n-1}(A, X^*)$. As in the above proof that χ is an epimorphism, there is a bounded approximate identity $\{e_\alpha\}$ in J and an R in $\mathscr{L}^{n-2}(A, X^*)$ such that

$$R(a_1, \cdots, a_{n-2}) = (-1)^n \lim S(a_1, \cdots, a_{n-2}, e_\alpha)$$

for all $a_1, \cdots, a_{n-2}$ in A. If a_{n-1} is in J and if $a_1, \cdots, a_{n-2}$ are in A, then

$$\begin{aligned}
&\delta^{n-1}R(a_1, \cdots, a_{n-1})\\
&= (-1)^n \lim\Big\{a_1 S(a_2, \cdots, a_{n-1}, e_\alpha) + \sum_{j=1}^{n-2}(-1)^j S(a_1, \cdots, a_j a_{j+1}, \cdots, a_{n-1}, e_\alpha)\\
&\qquad\qquad + (-1)^{n-1}S(a_1, \cdots, a_{n-2}, e_\alpha)a_{n-1}\Big\}\\
&= (-1)^n \lim\{\delta^n S(a_1, \cdots, a_{n-1}, e_\alpha) + (-1)^n S(a_1, \cdots, a_{n-2}, a_{n-1}e_\alpha)\}
\end{aligned}$$

because a_{n-1} and e_α are in J, which annihilates X^*. Since $\delta^n S=\psi_n T$, it follows that $\delta^n S(a_1, \cdots, a_{n-1}, e_\alpha)=0$ because e_α is in J. This completes the proof of the lemma.

Proof of Theorem 1. The definitions of θ and δ^n imply that $\delta^n\theta=\theta\delta^n$. Thus the mapping Q, defined in the statement of the theorem, is a well defined homomorphism from $\mathscr{H}^n(A/J, X^*)$ into $\mathscr{H}^n(A, X^*)$. We shall prove that Q is an isomorphism by induction on n over all Banach A-modules that are annihilated by J.

Now we consider $n=1$. If f is in X^*, then $\delta^1(f)(a)=af-fa=(a+J)f-f(a+J)=\delta^1(f)(a+J)=(\theta\delta^1(f))(a)$ by definition of δ^1, so that $\theta \operatorname{Im}\delta^1=\operatorname{Im}\delta^1$. If D is in $\operatorname{Ker}\delta^2$, which is contained in $\mathscr{L}^1(A, X^*)$, then $D(ab)=D(a)b+aD(b)$ for all a, b in A by definition of δ^2. If c is in J, then by Cohen's Factorization Theorem [1] we have $c=ab$ for some a and b in J. Thus $D(c)=aD(b)+D(a)b=0$ because J annihilates X^*. We may now define an operator T in $\mathscr{L}^1(A/J, X^*)$ by $T(a+J)=D(a)$ for all a in A. Then $\theta\ \dot{} = D$, and $\delta^2T=0$. This shows that Q is an isomorphism for $n=1$.

Suppose the result has been proved for n. We firstly observe that $\mathscr{L}^1(A/J, X^*)$ is, as a Banach A-module, the dual of the Banach A-module $Y=(A/J)\hat{\otimes}X$, the projective tensor product of Banach spaces [5, §1], where we define the module operations on generating elements $(a+J)\hat{\otimes}x$ of the tensor product by

$$(7)\qquad \begin{aligned} b((a+J)\hat{\otimes}x) &= (ba+J)\hat{\otimes}x-(b+J)\hat{\otimes}ax \quad \text{and} \\ ((a+J)\hat{\otimes}x)b &= (a+J)\hat{\otimes}xb \end{aligned}$$

and lift the definitions to Y by linearity and continuity. Because J annihilates X, equations (7) imply that J annihilates the A-module Y. Now by Lemma 2, our inductive hypothesis on n, and the reduction of dimension lemma for cohomology [5, 1(a)], the following isomorphisms hold:

$$\begin{aligned} \mathscr{H}^{n+1}(A, X^*) &\cong \mathscr{H}^n(A, \mathscr{L}^1(A/J, X^*)) \cong \mathscr{H}^n(A/J, \mathscr{L}^1(A/J, X^*)) \\ &\cong \mathscr{H}^{n+1}(A/J, X^*). \end{aligned}$$

Each of these isomorphisms is the natural one arising from the quotient A/J. Thus Q is an isomorphism for $n+1$. This completes the proof.

Our corollary generalizes [4, Theorems 4.1 and 4.2] from $n=1$ and 2 to any positive integer n.

Corollary 3. *Let A be a Banach algebra in which each closed cofinite ideal has a bounded approximate identity. If X is a finite dimensional Banach A-module and n is a positive integer, then* $\mathscr{H}^n(A, X)=\{0\}$.

Proof. The annihilator J of X is a closed cofinite ideal in A, and X is the dual of the Banach A-module X^*. By Theorem 1, we have $\mathscr{H}^n(A, X)$ isomorphic to $\mathscr{H}^n(A/J, X)$. An ideal in A/J is of the form I/J, where I is a closed cofinite ideal in A containing J. Since I has a bounded approximate identity, I^2 is equal to I by Cohen's Factorization Theorem [1]. This shows that A/J is a finite dimensional semisimple algebra. As every n-linear operator from A/J into X is continuous, $\mathscr{H}^n(A/J, X)$ coincides with Hochschild's cohomology groups [3] for the A/J-module X. Hochschild's nth-cohomology group for the A/J-module X is null [3, Theorem 4.1], and so $\mathscr{H}^n(A, X)=\{0\}$.

Remark 4. We now outline an example which shows that some assumption on J like that of a bounded approximate identity is necessary if the conclusion of Theorem 1 is to hold. Let X be a (finite dimensional) Banach space, and let X have the zero product ($xy=0$ for all x, y in X). Let A be the Banach algebra obtained by adjoining an identity to X, and let the ideal J be X. We regard X as an A-module with the natural module operations. Then A/J is equal to $C1$, and so $\mathscr{H}^1(A/J, X^*)$ is zero as may be proved in a number of ways (for example [3, Theorem 4.1]). However for the algebra A we obtain Im δ^1 is $\{0\}$, and Ker δ^2 is $\mathscr{L}^1(J, X^*)$, so that $\mathscr{H}^1(A, X^*)=\mathscr{L}^1(J, X^*)$ and the conclusion of Theorem 1 does not hold.

References

1. P. J. Cohen, *Factorization in group algebras*, Duke Math. J. **26** (1959), 199–205. MR **21** #3729.

2. J. Dixmier, *Les C*-algèbres et leurs représentations*, Cahiers Scientifiques, fasc. 29, Gauthier-Villars, Paris, 1964. MR **30** #1404.

3. G. Hochschild, *On the cohomology groups of an associative algebra*, Ann. of Math. (2) **46** (1945), 58–67. MR **6**, 114.

4. B. E. Johnson, *The Wedderburn decomposition of Banach algebras with finite dimensional radical*, Amer. J. Math. **90** (1968), 866–876. MR **38** #1529.

5. ———, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. (to appear).

Department of Mathematics, University of Edinburgh, Edinburgh, Scotland

Department of Mathematics, University of The Witwatersrand, Johannesburg, South Africa (Current address)