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COMMON FIXED POINT THEOREMS FOR ALMOST WEAKLY PERIODIC NONEXPANSIVE MAPPINGS

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ABSTRACT. The notions of normal structure, (convex) diminishing orbital diameters, regular orbital diameters (r.o.d.) have been generalized into a Hausdorff locally convex space (X, τ) whose topology τ is generated by a family $\mathscr{P}$ of seminorms. THEOREM 1. *Let $K \subseteq X$ be nonempty weakly compact convex with normal structure w.r.t. $\mathscr{P}$ and $\mathscr{F}$ be a (not necessarily finite nor commuting) family of almost weakly periodic nonexpansive mappings w.r.t. $\mathscr{P}$ on K. Then $\mathscr{F}$ has a common fixed point.* THEOREM 2. *Let $K \subseteq X$ be nonempty weakly compact convex and $\mathscr{F}$ be a semigroup with identity of almost weakly periodic nonexpansive mappings w.r.t. $\mathscr{P}$ on K. If $\mathscr{F}$ has r.o.d. w.r.t. $\mathscr{P}$, then $\mathscr{F}$ has a common fixed point.* COROLLARY. *If $K \subseteq X$ is nonempty weakly compact convex and $\mathscr{F} = \{f_1, \cdots, f_n\}$ is a finite commuting family of pointwise periodic nonexpansive mappings w.r.t. $\mathscr{P}$ on K, then $\mathscr{F}$ has a common fixed point.*

1. **Preliminaries and notations.** We shall denote by $\mathscr{Z}^+$ the set of all nonnegative integers, $\mathscr{N}$ the set of all natural numbers. If X is a nonempty set and $f, g: X \to X$, we denote $f^0 = I$, the identity mapping on X, $fg = f \circ g$, the composition of f and g, and $f^{n+1} = f(f^n)$, for each $n \in \mathscr{Z}^+$. If $x \in X$ and $n \in \mathscr{Z}^+$, we denote $O(f, n, x) = \{f^{n+k}(x) : k \in \mathscr{Z}^+\}$. If $\mathscr{F}$ is a family of mappings on X, for each $x \in X$ and each $f \in \mathscr{F}$, we denote $\mathscr{F}(x) = \{g(x) : g \in \mathscr{F}\}$ and $\mathscr{F}f(x) = \{gf(x) : g \in \mathscr{F}\}$. If d is a pseudometric or a seminorm on X, and $A \subseteq X$ is nonempty, $d(A)$ denotes the diameter of A w.r.t. d. If (*) is a property, then a mapping f or a family $\mathscr{F}$ of mappings on X is said to have (*) iff f or $\mathscr{F}$ has (*) at x for each $x \in X$. If (X, τ) is a topological vector space and $A \subseteq X$, we denote by $\mathrm{Co}(A)$ the convex hull of A and $\bar{\mathrm{C}}\mathrm{o}(A)$ the closed convex hull of A.

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DEFINITION 1.1. Let (X, d) be a metric space and $f: X \to X$. Then f is said to have diminishing orbital diameters at $x \in X$ iff either $f(x)=x$ or there exists an $n \in \mathcal{N}$ such that $d(O(f, n, x)) < d(O(f, 0, x))$.

The above notion was first introduced by L. P. Belluce and W. A. Kirk in [**1**]. The following are some of its generalizations.

DEFINITION 1.2. Let (X, τ) be a Hausdorff (completely regular) space whose topology τ is generated by a family $\{d_\lambda\}_{\lambda \in \Lambda}$ of pseudometrics on X. (i) If $f: X \to X$, then f is said to have diminishing orbital diameters (d.o.d.) w.r.t. $\{d_\lambda\}_{\lambda \in \Lambda}$ at $x \in X$ iff either $f(x)=x$ or there exist a $\lambda \in \Lambda$ and an $n \in \mathcal{N}$ such that $d_\lambda(O(f, n, x)) < d_\lambda(O(f, 0, x))$. (ii) If $\mathscr{F}$ is a semigroup (under composition) with identity of mappings on X, then $\mathscr{F}$ is said to have d.o.d. w.r.t. $\{d_\lambda\}_{\lambda \in \Lambda}$ at $x \in X$ iff either $\mathscr{F}(x)=\{x\}$ or there exist a $\lambda \in \Lambda$ and an $f \in \mathscr{F}$ such that $d_\lambda(\mathscr{F}f(x)) < d_\lambda(\mathscr{F}(x)) < \infty$.

Definition 1.2(i) is introduced by the author in [**5**]. In case $\{d_\lambda\}_{\lambda \in \Lambda}$ contains only a single metric, Definition 1.2 (ii) is introduced by M. T. Kiang in [**4**]. If the domain under consideration is convex, we have the following:

DEFINITION 1.3. Let (X, τ) be a Hausdorff locally convex space whose topology τ is generated by a family $\mathscr{P}$ of seminorms on X, $K \subseteq X$ be nonempty closed convex and $\mathscr{F}$ be a semigroup with identity of mappings on K. For each $x \in K$, let $C(\mathscr{F}, x)$ be the smallest closed convex subset of K containing x which is $\mathscr{F}$-invariant (i.e. invariant under each $f \in \mathscr{F}$). Then (i) $\mathscr{F}$ is said to have convex diminishing orbital diameters (c.d.o.d.) w.r.t. $\mathscr{P}$ at $x \in K$ iff either $\mathscr{F}(x)=\{x\}$ or there exist a $p \in \mathscr{P}$ and an $x_0 \in \overline{\text{Co}}(\mathscr{F}(x))$ such that $p(\mathscr{F}(x_0)) < p(\mathscr{F}(x)) < \infty$; (ii) $\mathscr{F}$ is said to have regular orbital diameters (r.o.d.) w.r.t. $\mathscr{P}$ at $x \in K$ iff either $\mathscr{F}(x)=\{x\}$ or there exist a $p \in \mathscr{P}$ and $y, z \in C(\mathscr{F}, x)$ such that $\sup\{p(y-f(z)): f \in \mathscr{F}\} < p(C(\mathscr{F}, x))$.

It is clear that $\mathscr{F}$ has d.o.d. w.r.t. $\mathscr{P}$ at x implies $\mathscr{F}$ has c.d.o.d. w.r.t. $\mathscr{P}$ at x which in turn implies $\mathscr{F}$ has r.o.d. w.r.t. $\mathscr{P}$ at x. In case $\mathscr{P}$ contains only a single norm, Definition 1.3(i) reduces to the notion of c.d.o.d. defined by M. T. Kiang in [**4**]. Also if $\mathscr{F}=\{f^n: n \in \mathscr{Z}^+\}$ is the semigroup with identity generated by a single mapping f on K, Definition 1.3(ii) reduces to the notion of r.o.d. for a single mapping defined by C. S. Wong in [7].

From now on (X, τ) denotes a Hausdorff locally convex space whose topology τ is generated by a family $\mathscr{P}$ of seminorms on X.

DEFINITION 1.4. $K \subseteq X$ is said to have normal structure w.r.t. $\mathscr{P}$ iff for any bounded convex subset H of K, if H contains more than one point, then there exist a $p \in \mathscr{P}$ and an $x_0 \in H$ such that $\sup\{p(x_0-x): x \in H\} < p(H)$.

The notion of normal structure was first introduced by M. S. Brodskiĭ and D. P. Milman in [**2**]. The above generalization was obtained independently by R. D. Holmes and A. Lau in [**3**] and by the author in [**6**]. It is known that:

LEMMA 1.5. *If $K \subseteq X$ is compact convex, then K has normal structure w.r.t. $\mathscr{P}$.*

If $K \subseteq X$ is nonempty closed convex and $\mathscr{F}$ is a family of mappings on K, it is easy to see that $\mathrm{Co}(\mathscr{F}(x))$ has normal structure w.r.t. $\mathscr{P}$ implies $\mathscr{F}$ has r.o.d. at x, for each $x \in K$.

2. Common fixed point theorems.

DEFINITION 2.1. Let $K \subseteq X$ be nonempty and $f: K \to K$. Then (i) f is almost weakly periodic iff $x \in \overline{\mathrm{Co}}(O(f, 1, x))$, for each $x \in K$; (ii) f is weakly periodic iff for each $x \in K$, there is a subsequence $\{f^{n_i}(x)\}_{i=1}^{\infty}$ of $\{f^n(x)\}_{n=1}^{\infty}$ such that $f^{n_i}(x) \to x$ weakly; (iii) f is almost pointwise periodic iff for each $x \in K$, $x \in \bigcap_{n=1}^{\infty} \overline{\mathrm{Co}}(O(f^n, 1, x))$; (iv) f is pointwise periodic iff for each $x \in K$, there is an $N(x) \in \mathscr{N}$ such that $f^N(x) = x$; (v) f is periodic iff there is an $N \in \mathscr{N}$ such that $f^N = I$, the identity mapping on K. (vi) f is nonexpansive w.r.t. $\mathscr{P}$ iff for each $p \in \mathscr{P}$, $p(f(x) - f(y)) \leqq p(x - y)$, for all $x, y \in K$.

Definition 2.1 (vi) was introduced independently by R. D. Holmes and A. Lau in [**3**] and by the author in [**6**].

It is clear that f is periodic implies f is pointwise periodic, f is pointwise periodic implies f is almost pointwise periodic and weakly periodic, and f is weakly periodic (respectively almost pointwise periodic) implies f is almost weakly periodic. Also f is nonexpansive w.r.t. $\mathscr{P}$ implies f is continuous.

THEOREM 2.2. *Let $K \subseteq X$ be nonempty weakly compact convex, $\mathscr{F}$ be a (not necessarily finite nor commuting) family of almost weakly periodic nonexpansive mappings w.r.t. $\mathscr{P}$ on K and $\mathscr{S}$ be the semigroup with identity generated by $\mathscr{F}$. Suppose for each $x \in K$, $\overline{\mathrm{Co}}(\mathscr{S}(x))$ has normal structure w.r.t. $\mathscr{P}$. Then $\mathscr{F}$ has a common fixed point.*

PROOF. By weak compactness of K and by Zorn's lemma, let K_1 be minimal w.r.t. being a nonempty closed convex subset of K which is $\mathscr{F}$-invariant. Suppose there exist an $r \in K_1$ and an $f_0 \in \mathscr{F}$ such that $f_0(x) \neq x$. Then $\mathscr{S}(x)$ and hence $\overline{\mathrm{Co}}(\mathscr{S}(x))$ contains more than one point. Thus there exist a $p \in \mathscr{P}$ and an $x_0 \in \overline{\mathrm{Co}}(\mathscr{S}(x))$ such that

$$r_0 = \sup\{p(x_0 - z) : z \in \overline{\mathrm{Co}}(\mathscr{S}(x))\} < p(\overline{\mathrm{Co}}(\mathscr{S}(x))).$$

Define

$$M = \{y \in K_1 : \sup\{p(y - z) : z \in \mathscr{S}(x)\} \leqq r_0\}.$$

Then $M\neq\varnothing$ since $x_0\in M$. It is also clear that M is closed and convex. We shall show that M is $\mathscr{F}$-invariant. Indeed let $f\in\mathscr{F}$ and $y\in M$. If $z\in\mathscr{S}(x)$, then $z\in\overline{\mathrm{Co}}(O(f,1,z))$ and so for any $\varepsilon>0$, there is a $z_1\in\mathrm{Co}(O(f,1,z))$ with $p(z-z_1)<\varepsilon$. But then $z_1=\sum_{i=1}^m\lambda_i f^{n_i}(z)$ for some $n_i\geqq 1$ and $0\leqq\lambda_i\leqq 1$ for each $i=1,\cdots,m$ with $\sum_{i=1}^m\lambda_i=1$. Since $f^{n_i-1}(z)\in\mathscr{S}(x)$ for each $i=1,\cdots,m$, we have $p(y-f^{n_i-1}(z))\leqq r_0$, for each $i=1,\cdots,m$. Thus for each $\varepsilon>0$,

$$\begin{aligned}p(f(y)-z)&\leqq p(f(y)-z_1)+p(z_1-z)<p\left(f(y)-\sum_{i=1}^m\lambda_i f^{n_i}(z)\right)+\varepsilon\\&\leqq\sum_{i=1}^m\lambda_i p(f(y)-f^{n_i}(z))+\varepsilon\leqq\sum_{i=1}^m\lambda_i p(y-f^{n_i-1}(z))+\varepsilon\\&\leqq\sum_{i=1}^m\lambda_i r_0+\varepsilon=r_0+\varepsilon,\end{aligned}$$

so that $p(f(y)-z)\leqq r_0$. Hence $f(y)\in M$, for each $f\in\mathscr{F}$ and each $y\in M$; i.e. M is $\mathscr{F}$-invariant. By minimality of K_1, we must have $M=K_1$. Since $r_0<p(\overline{\mathrm{Co}}(\mathscr{S}(x))$ there exist $a,b\in\overline{\mathrm{Co}}(\mathscr{S}(x))$ such that $p(a-b)>r_0$. But then either $a\notin M$ or $b\notin M$, which is a contradiction. Therefore we must have $f(x)=x$ for each $f\in\mathscr{F}$ and each $x\in K_1$. Since $K_1\neq\varnothing$, $\mathscr{F}$ has a common fixed point.

In case $\mathscr{F}$ is a semigroup with identity, we have the following slight generalization of the above theorem:

THEOREM 2.3. *Let $K\subseteq X$ be nonempty weakly compact convex and $\mathscr{F}$ be a semigroup with identity of almost weakly periodic nonexpansive mappings w.r.t. $\mathscr{P}$ on K. If $\mathscr{F}$ has r.o.d. w.r.t. $\mathscr{P}$, then $\mathscr{F}$ has a common fixed point.*

PROOF. By weak compactness of K and by Zorn's lemma, let K_1 be minimal w.r.t. being a nonempty closed convex subset of K which is $\mathscr{F}$-invariant. Suppose there exist an $x_0\in K_1$ and an $f_0\in\mathscr{F}$ such that $f_0(x_0)\neq x_0$. By minimality of K_1, we must have $C(\mathscr{F},x_0)=K_1$. Since $\mathscr{F}$ has r.o.d. w.r.t. $\mathscr{P}$ and $\mathscr{F}(x_0)\neq\{x_0\}$, there exist $y_0,z_0\in C(\mathscr{F},x_0)$ and a $p\in\mathscr{P}$ with $r_0=\sup\{p(y_0-f(z_0)):f\in\mathscr{F}\}<p(C(\mathscr{F},x_0))$. Define

$$M=\{y\in K_1:\sup\{p(y-f(z_0)):f\in\mathscr{F}\}\leqq r_0\}.$$

Then M is nonempty as $y_0\in M$. It is clear that M is also closed and convex. Also M can be shown to be $\mathscr{F}$-invariant by the same argument as in the proof of Theorem 2.2. Hence $M=K_1$, by the minimality of K_1. Next we define $N=\{z\in K_1:p(y-f(z))\leqq r_0$, for all $y\in K_1$ and all $f\in\mathscr{F}\}$. Then N is nonempty as $z_0\in N$. Since each $f\in\mathscr{F}$ is continuous, N is closed. Since $\mathscr{F}$ is a semigroup, N is $\mathscr{F}$-invariant. We shall now show that N is also convex.

Indeed, let $z_1, z_2 \in N$, $0<\lambda<1$, and $z=\lambda z_1+(1-\lambda)z_2$. Suppose $y \in K_1$ and $f \in \mathscr{F}$. Since $y \in \overline{\mathrm{Co}}(O(f, 1, y))$, for each $\varepsilon>0$, there is a $y_1 \in \mathrm{Co}(O(f, 1, y))$ with $p(y-y_1)<\varepsilon$. But then $y_1=\sum_{i=1}^m \lambda_i f^{n_i}(y)$, for some $n_i \geqq 1$, $0 \leqq \lambda_i \leqq 1$, for each $i=1, \cdots, m$ with $\sum_{i=1}^m \lambda_i=1$. Thus for each $\varepsilon>0$,

$$\begin{aligned}
p(y-f(z)) \leqq p(y-y_1)+p(y_1-f(z)) &< \varepsilon + p\left(\sum_{i=1}^m \lambda_i f^{n_i}(y)-f(z)\right) \\
&\leqq \varepsilon + \sum_{i=1}^m \lambda_i p(f^{n_i}(y)-f(z)) \\
&\leqq \varepsilon + \sum_{i=1}^m \lambda_i p(f^{n_i-1}(y)-(\lambda z_1+(1-\lambda)z_2)) \\
&\leqq \varepsilon + \sum_{i=1}^m \lambda_i [\lambda p(f^{n_i-1}(y)-z_1)+(1-\lambda)p(f^{n_i-1}(y)-z_2)] \\
&\leqq \varepsilon + \sum_{i=1}^m \lambda_i [\lambda r_0+(1-\lambda)r_0] = \varepsilon + \sum_{i=1}^m \lambda_i r_0 = \varepsilon + r_0,
\end{aligned}$$

so that $p(y-f(z)) \leqq r_0$, for all $y \in K_1$ and all $f \in \mathscr{F}$. Hence $z=\lambda z_1+(1-\lambda)z_2$ is in N and therefore N is convex. By minimality of K_1 again, $K_1=N$. Since $r_0<p(C(\mathscr{F}, x_0))=p(K_1)$, there are $a, b \in K_1$ with $p(a-b)>r_0$. Since $I \in \mathscr{F}$, it follows that neither a nor b is in N, which is a contradiction. Therefore $f(x)=x$ for each $x \in K_1$ and each $f \in \mathscr{F}$. Since K_1 is nonempty, $\mathscr{F}$ has a common fixed point.

Corollary 2.4. *Let $K \subseteq X$ be nonempty weakly compact convex with normal structure w.r.t. $\mathscr{P}$ and $\mathscr{F}$ be any (not necessarily finite nor commuting) family of almost weakly periodic nonexpansive mappings w.r.t. $\mathscr{P}$ on K. Then $\mathscr{F}$ has a common fixed point.*

Corollary 2.5. *Let $K \subseteq X$ be nonempty weakly compact convex and $\mathscr{F}$ be a semigroup with identity of almost weakly periodic nonexpansive mappings w.r.t. $\mathscr{P}$ on K. If $\mathscr{F}$ has c.d.o.d. w.r.t. $\mathscr{P}$ then $\mathscr{F}$ has a common fixed point.*

Corollary 2.6. *Let $K \subseteq X$ be nonempty weakly compact convex and $\mathscr{F}=\{f_1, \cdots, f_n\}$ be a finite commuting family of pointwise periodic nonexpansive mappings w.r.t. $\mathscr{P}$ on K. Then $\mathscr{F}$ has a common fixed point.*

Proof. Let $\mathscr{S}$ be the semigroup with identity generated by $\mathscr{F}$. Then for each $x \in K$, $\mathscr{S}(x)$ is a finite subset of K and so $\overline{\mathrm{Co}}(\mathscr{S}(x))$ is compact convex. By Lemma 1.5, $\overline{\mathrm{Co}}(\mathscr{S}(x))$ has normal structure w.r.t. $\mathscr{P}$ (and hence r.o.d. w.r.t. $\mathscr{P}$) so that by Theorem 2.2 (or by Theorem 2.3), $\mathscr{F}$ has a common fixed point.

COROLLARY 2.7. *Let $K \subseteq X$ be nonempty compact convex and $\mathscr{F}$ be any (not necessarily finite nor commuting) family of almost weakly periodic nonexpansive mappings w.r.t. $\mathscr{P}$ on K. Then $\mathscr{F}$ has a common fixed point.*

Since every uniformly convex Banach space (in particular, every Hilbert space) has normal structure, we have the following observation: if K is any nonempty weakly compact (or equivalently bounded closed) convex subset of a uniformly convex Banach space X, and if there is an almost weakly periodic nonexpansive mapping on K which has a unique fixed point $x_0 \in K$, then x_0 is fixed under any other almost weakly periodic (respectively weakly periodic, almost pointwise periodic, pointwise periodic or periodic) nonexpansive mapping on K. In particular, if K is the closed unit ball $\{x \in X: \|x\| \leqq 1\}$, then $f(0)=0$, for any almost weakly periodic nonexpansive mapping f on K. Indeed, define $f_0(x)=-x$, for each $x \in K$, then f_0 is periodic and isometric (and hence nonexpansive), and 0 is the unique fixed point of f_0.

REMARK 2.8. Even when K is a nonempty weakly compact convex subset of a Banach space X, it is not known whether a countably infinite commuting family $\mathscr{F}$ of periodic nonexpansive mappings on K will have a common fixed point.

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