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ON THE LOWER BOUND OF THE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION WITH INFINITE VARIANCE

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ABSTRACT. Let N_n be the number of real roots of a random algebraic equation $\sum_0^n \xi_v x^v = 0$, where the coefficients ξ_v's are independent random variables with common characteristic function $\exp(-C|t|^\alpha)$, C being a positive constant and $\alpha \geqq 1$. It is proved that

$$N_n \geqq (\mu \log n)/(\log \log n).$$

The measure of the exceptional set tends to zero as n tends to infinity.

1. **Introduction.** Consider the algebraic equation

$$f(x) = \xi_0 + \xi_1 x + \cdots + \xi_n x^n = 0 \tag{1}$$

where the ξ_v's are independent random variables assuming real values only, and let N_n be the number of real roots of (1).

The problem of finding a lower bound of N_n has been considered by several authors. But none has taken up the case when the coefficients have infinite variance. Samal [3] has considered the general case when the ξ_v's have identical distribution, with expectation zero and variance, and third absolute moment finite and nonzero.

(2) The object of this paper is to find a lower bound for N_n when the ξ_v's are identically distributed with a common characteristic function $\exp(-C|t|^\alpha)$, where C is a positive constant and $\alpha \geqq 1$. For $1 \leqq \alpha < 2$ this represents a symmetric stable distribution with infinite variance.

A stronger result has been obtained by Littlewood and Offord [2] in the case $\alpha = 2$. They have proved that $N_n \geqq (\mu \log n)/(\log \log \log n)$ except for a set of measure which tends to zero as n tends to infinity.

Although our result is weaker than the result obtained by them, it holds for all $\alpha \geqq 1$. The importance of our result lies in the range $1 < \alpha < 2$ when the variance is infinite.

Our main theorem is the following:

THEOREM. *Let $f(x)$ be a polynomial of degree n chosen at random with condition* (2). *Then for $n \geqq n_0$, the number of real roots of most of the equations*

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$f(x)=0$ is at least $(\mu \log n)/(\log\log n)$. The measure of the exceptional set tends to zero as n tends to infinity.

We shall always suppose that any inequality is satisfied when n is large. We shall denote positive constants by μ.

2. The following lemma is necessary in the beginning of the proof.

LEMMA 1. *If a random variable $\xi(u)$ has characteristic function $\exp(-C\,|t|^{\alpha})$ then*

$$\Pr(|\xi(u)| > \varepsilon) < \frac{2^{1+\alpha}\cdot C}{1+\alpha}\cdot\frac{1}{\varepsilon^{\alpha}}$$

for every $\varepsilon>0$.

PROOF. Let $F(x)$ be the distribution function of $\xi(u)$. Then, by Gnedenko and Kolmogorov [**1**, p. 53],

$$\begin{aligned}\Pr(|\xi(u)| > \varepsilon) &= 1 - \{F(\varepsilon) - F(-\varepsilon)\}\\ &\leqq 1 - \left\{\frac{1}{2}\,\varepsilon\left|\int_{-2/\varepsilon}^{2/\varepsilon}\exp(-C\,|t|^{\alpha})\,dt\right| - 1\right\}\\ &= 2 - \frac{1}{2}\,\varepsilon\int_{-2/\varepsilon}^{2/\varepsilon}\exp(-C\,|t|^{\alpha})\,dt\\ &= 2 - \varepsilon\int_{0}^{2/\varepsilon}\exp(-C\,|t|^{\alpha})\,dt\\ &\leqq 2 - \varepsilon\int_{0}^{2/\varepsilon}(1 - Ct^{\alpha})\\ &= \frac{\varepsilon C}{1+\alpha}\,(2/\varepsilon)^{1+\alpha} = \frac{2^{1+\alpha}\cdot C}{1+\alpha}\cdot\frac{1}{\varepsilon^{1+\alpha}}.\end{aligned}$$

3. **Proof of the Theorem.** We shall follow the line of proof of Samal [**3**] with modifications where necessary.

(3) Take constants A and B such that $0<B<1$ and $A>1$ and let $\lambda=\log n$.

Let M be the integer defined by

$$M = [2^{1+\alpha}Ae\lambda^{\alpha}/B] + 1 \tag{4}$$

where $[x]$ is the integral part of x; and let k be determined by

$$M^{2k} \leqq n < M^{2k+2}. \tag{5}$$

We consider the polynomial $f=\sum_{v=0}^{n}\xi_v x^v$ at the points

$$x_m = \left(1 - \frac{1}{M^{2m}}\right)^{1/\alpha} \qquad (m = [\tfrac{1}{2}k] + 1,\ [\tfrac{1}{2}k] + 2, \cdots, k). \tag{6}$$

There are $\frac{1}{2}k$ points if k is even and $\frac{1}{2}(k+1)$ points if k is odd. We have

$$f(x_m) = U_m + R_m \tag{7}$$

where

$$U_m = \sum_{M^{2m-1}+1}^{M^{2m+1}} \xi_v x_m^v \quad \text{and} \quad R_m = \left(\sum_{0}^{M^{2m-1}} + \sum_{M^{2m+1}+1}^{n}\right) \xi_v x_m^v. \tag{8}$$

Obviously U_m and U_{m+1} are independent random variables. We also define

$$V_m = \left(\sum_{M^{2m-1}+1}^{M^{2m+1}} x_m^{\alpha v}\right)^{1/\alpha}. \tag{9}$$

We proceed to estimate

$$\begin{aligned} P &= \Pr\{(U_{2m} > V_{2m}, U_{2m+1} < -V_{2m+1}) \cup (U_{2m} < -V_{2m}, U_{2m+1} > V_{2m+1})\} \\ &= \Pr(U_{2m} > V_{2m})\Pr(U_{2m+1} < -V_{2m+1}) \\ &\quad + \Pr(U_{2m} < -V_{2m})\Pr(U_{2m+1} > V_{2m+1}). \end{aligned}$$

Since the charactersitic function of $\xi_v(u)$ is $\exp(-C\,|t|^\alpha)$ therefore the characteristic function of U_m is

$$\exp\left\{-C\,|t|^\alpha \sum_{M^{2m-1}+1}^{M^{2m+1}} x_m^{\alpha v}\right\} = \exp\{-C\,|t|^\alpha V_m^\alpha\}.$$

Hence the characteristic function of U_m/V_m is $\exp\{-C\,|t|^\alpha\}$ which is independent of m.

If $F(x)$ is the distribution function of U_m/V_m for all m, then

$$\begin{aligned} P &= \{1 - F(1)\}F(-1) + F(-1)\{1 - F(1)\} \\ &= 2F(-1)\{1 - F(1)\} = \delta \quad \text{(say)}. \end{aligned}$$

Obviously $\delta > 0$.

We define E_m and F_m as follows:

$$\begin{aligned} E_m &= \{U_{2m} > V_{2m},\ U_{2m+1} < -V_{2m+1}\}, \\ F_m &= \{U_{2m} < -V_{2m},\ U_{2m+1} > V_{2m+1}\}. \end{aligned}$$

We have shown above that $m(E_m \cup F_m) = \delta$.

Let η_m be a random variable such that it takes value 1 on $E_m \cup F_m$ and zero elsewhere. In other words,

$$\begin{aligned} \eta_m &= 1 \quad \text{with probabiity } \delta, \\ &= 0 \quad \text{with probability } 1 - \delta. \end{aligned}$$

It follows as in Samal [3, p. 439] that outside a set of measure at most μ/k, either $U_{2m} > V_{2m}$ and $U_{2m+1} < -V_{2m+1}$ or $U_{2m} < -V_{2m}$ and $U_{2m+1} > V_{2m+1}$ for at least μk values of m.

4. Before we consider R_m we need the following lemmas.

LEMMA 2.

$$\left|\sum_{M^{2m+1}+1}^{n} \xi_v(u)x_m^v\right| < \tfrac{1}{2}V_m$$

except for a set of measure at most $\{2^{1+2\alpha}\cdot C/(1+\alpha)\}(Ae/B)\, e^{-M}$.

PROOF. The characteristic function of $\sum_{M^{2m+1}+1}^{n} \xi_v(u)x_m^v$ is

$$\exp\left\{-C\,|t|^\alpha \sum_{M^{2m+1}+1}^{n} x_m^{\alpha v}\right\}.$$

Hence by using Lemma 1,

$$\begin{aligned} P_1 &= \Pr\left\{\left|\sum_{M^{2m+1}+1}^{n} \xi_v(u)x_m^v\right| \geqq \tfrac{1}{2}V_m\right\} \\ &\leqq \left\{2^{1+\alpha}\cdot C \sum_{M^{2m+1}+1}^{n} x_m^{\alpha v}\right\}\Big/(1+\alpha)(\tfrac{1}{2}V_m)^\alpha \\ &= \left(C\cdot 2^{1+2\alpha} \sum_{M^{2m+1}+1}^{n} x_m^{\alpha v}\right)\Big/(1+\alpha)V_m^\alpha. \end{aligned}$$

But

$$\begin{aligned} \sum_{M^{2m+1}+1}^{n} x_m^{\alpha v} &\leqq \frac{x_m^{\alpha(M^{2m+1}+1)}}{1-x_m^\alpha} = M^{2m}\left(1-\frac{1}{M^{2m}}\right)^{M^{2m+1}+1} \\ &\leqq M^{2m}\left(1-\frac{1}{M^{2m}}\right)^{M^{2m+1}+1} \\ &\leqq M^{2m}\left(1-\frac{1}{M^{2m}}\right)^{M^{2m+1}} \leqq M^{2m}e^{-M}. \end{aligned}$$

Also it follows as in Samal [3, p. 439] that

$$\begin{aligned} V_m^\alpha &= \sum_{M^{2m-1}+1}^{M^{2m+1}} x_m^{\alpha v} \geqq \sum_{M^{2m-1}+1}^{M^{2m}} x_m^{\alpha v} \\ &\geqq (M^{2m}-M^{2m-1})\left(1-\frac{1}{M^{2m}}\right)^{M^{2m}} \geqq M^{2m}B\cdot\frac{e^{-1}}{A}. \end{aligned} \tag{10}$$

Therefore

$$\begin{aligned} P_1 &\leqq \frac{C\cdot 2^{1+2\alpha}}{1+\alpha}\frac{M^{2m}e^{-M}}{M^{2m}B(e^{-1}/A)} \\ &= \frac{2^{1+2\alpha}\cdot C}{1+\alpha}\frac{Ae}{B}\,e^{-M}. \end{aligned}$$

LEMMA 3.

$$\left|\sum_{0}^{M^{2m-1}} \xi_v(u)x_m^v\right| < \lambda\left(\sum_{0}^{M^{2m-1}} x_m^{\alpha v}\right)^{1/\alpha}$$

except for a set of measure at most $2^{1+\alpha}\cdot C/(1+\alpha)\lambda^\alpha$.

PROOF. The characteristic function of $\sum_0^{M^{2m-1}} \xi_v(u)x_m^v$ is

$$\exp\left\{-C\,|t|^\alpha \sum_{0}^{M^{2m-1}} x_m^{\alpha v}\right\}.$$

Therefore, by using Lemma 1,

$$P_2 = \Pr\left\{\left|\sum_{0}^{M^{2m-1}} \xi_v(u)x_m^v\right| \geqq \lambda\left(\sum_{0}^{M^{2m-1}} x_m^{\alpha v}\right)^{1/\alpha}\right\} \leqq 2^{1+\alpha}\cdot C/(1+\alpha)\lambda^\alpha.$$

Thus by using Lemmas 2 and 3, we have, for any given m,

$$|R_m| < \tfrac{1}{2}V_m + \lambda\left(\sum_{0}^{M^{2m-1}} x_m^{\alpha v}\right)^{1/\alpha}$$

except for a set of measure at most $\mu_1\, e^{-M}+\mu_2/\lambda^\alpha$ where μ_1 and μ_2 are positive constants which can be determined in terms of the initial constants C, α, A and B. But

$$\begin{aligned}
\lambda\left(\sum_{0}^{M^{2m-1}} x_m^{\alpha v}\right)^{1/\alpha} &< \lambda(M^{2m-1}+1)^{1/\alpha}\\
&\leqq \lambda M^{(2m-1)/\alpha}\left(1+\frac{1}{M^{2m-1}}\right)^{1/\alpha}\\
&\leqq 2^{1/\alpha}\lambda M^{(2m-1)/\alpha}\\
&= 2^{1/\alpha}\lambda M^{2m/\alpha}/M^{1/\alpha}\\
&\leqq 2^{1/\alpha}\lambda\left(\frac{Ae}{B}\right)^{1/\alpha} V_m/M^{1/\alpha}\\
&= \lambda\left(\frac{2Ae}{BM}\right)^{1/\alpha} V_m.
\end{aligned}$$

The last steps above follow from (10). Hence by using (4), it follows that

$$|R_m| < V_m$$

for $m=[\frac{1}{2}k]+1,\ [\frac{1}{2}k]+2, \cdots, k$. Thus it follows from (7) that except for a set of measure at most

$$(\mu/k) + \mu_1\cdot\tfrac{1}{2}(k+1)e^{-M} + \mu_2\cdot\tfrac{1}{2}(k+1)/\lambda^\alpha. \tag{11}$$

f shows a change of sign and therefore has a root, between x_{2m} and x_{2m+1} for at least μk values of m.

It can be easily shown by using (4) and (5) that

$$\mu_1(\log n)^\alpha \leqq M \leqq \mu_2(\log n)^\alpha$$

and

$$(\mu_3 \log n)/(\log\log n) \leqq k \leqq (\mu_4 \log n)/(\log\log n).$$

We shall complete the proof of the theorem by showing that the measure of the exceptional set tends to zero as n tends to infinity.

$$\begin{aligned} ke^{-M} &\leqq \{(\mu_4 \log n)/(\log\log n)\}\exp\{-\mu_1(\log n)^\alpha\} \\ &= (\mu_4/\log\log n)\cdot\{(\log n)/\exp[\mu_1(\log n)^\alpha]\} \end{aligned}$$

which tends to zero since $\alpha \geqq 1$. Again

$$\begin{aligned} k/\lambda^\alpha &\leqq \{(\mu_4 \log n)/(\log\log n)\}/(\log n)^\alpha \\ &= \{\mu_4/(\log\log n)\}/(\log n)^{\alpha-1} \end{aligned}$$

which tends to zero for the same reason as above. Also $1/k$ tends to zero as η tends to infinity. Hence (11) tends to zero as n tends to infinity. Obviously, the measure of the exceptional set is $O(1/(\log\log n)(\log n)^{\alpha-1})$ if $1 \leqq \alpha < 2$ and $O(\log\log n/\log n)$ if $\alpha \geqq 2$, when n tends to infinity.

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