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ON OSCILLATIONS FOR SOLUTIONS OF nTH ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. Necessary and sufficient conditions are given that all solutions of $x^{(n)}+f(t, x, x', \cdots, x^{(n-2)})=0$ are oscillatory for n even and are oscillatory or tend monotonically to zero as $t\to\infty$ for n odd. The results generalize recent results of J. S. W. Wong and G. H. Ryder and D. V. V. Wend.

1. **Introduction.** In this paper we are dealing with differential equations of the form

$$(*) \qquad x^{(n)} + f(t, x, x', \cdots, x^{(n-2)}) = 0,$$

where f is continuous in $[a, \infty)\times R^{n-1}$, $a\geqq 0$.

We consider only nontrivial solutions of $(*)$ which are indefinitely continuable to the right. A solution of $(*)$ is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually of constant sign. The equation $(*)$ is said to be oscillatory if every solution of $(*)$ is oscillatory. Recently, the present author **[1]** gave a definition, called generalized strongly continuous, which is a generalization of Wong's **[4]** definition and discussed the oscillatory properties of $(*)$. A function $f(t, x_1, \cdots, x_{n-1})$ is called generalized strongly continuous from the left at x_{1c} if $f(t, x_1, \cdots, x_{n-1})$ is continuous in $[a, \infty)\times R^{n-1}$, $a\geqq 0$, and for each $\varepsilon>0$ there exists $\delta>0$, $T\geqq 0$ and $x_\delta\in[x_{1c}-\delta, x_{1c}]$ such that for all $x_1\in[x_{1c}-\delta, x_{1c}]$, for all x_i satisfying $|x_i-k_i|\leqq\delta$ (k_i are arbitrary real constants) for $i=2, \cdots, n-1$, and for all $t\geqq T$,

$$\begin{aligned}(1-\varepsilon)f(t, x_\delta, k_2, \cdots, k_{n-1}) &\leqq f(t, x_1, \cdots, x_{n-1}) \\ &\leqq (1+\varepsilon)f(t, x_{1c}, k_2, \cdots, k_{n-1}).\end{aligned}$$

Generalized strong continuity from the right at x_{1c} is defined analogously. A function $f(t, x_1, \cdots, x_{n-1})$ is said to be generalized strongly continuous at x_{1c} if it is generalized strongly continuous both from the left and from the right at x_{1c}.

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The purpose of this paper is to extend Wong's [4, Theorem 4] result to the arbitrary nth order equation $(*)$. The results of this paper also extend recent results of Ryder and Wend [3].

For convenience in stating our theorems, we list the following conditions:

(1) $$x_1 f(t, x_1, \cdots, x_{n-1}) \geqq 0 \qquad (x_1 \neq 0),$$

(2) $$x^{(n)} + f(t, x, x', \cdots, x^{(n-2)}, x^{(n-1)}) = 0,$$
where f is continuous in $[a, \infty) \times R^n$, $a \geqq 0$,

(3) there exists constants c $(\neq 0)$ and $k_2, \cdots, k_{n-1}$ such that

$$\left| \int^{\infty} t^{n-1} f(t, c, k_2, \cdots, k_{n-1})\, dt \right| < \infty.$$

2. Nonoscillation theorems.

THEOREM A [1]. *Assume that n is even and that condition* (1) *holds. Let* $f(t, x_1, \cdots, x_{n-1})$ *be generalized strongly continuous from the left for* $x_1 > 0$ *and generalized strongly continuous from the right for* $x_1 < 0$. *Then condition* (3) *is a necessary and sufficient condition for equation* $(*)$ *to have a bounded nonoscillatory solution.*

THEOREM 1. *Assume that n is odd and that condition* (1) *holds. Let* $f(t, x_1, \cdots, x_{n-1})$ *be generalized strongly continuous.*

Then condition (3) *is a necessary and sufficient condition for equation* $(*)$ *to have a bounded nonoscillatory solution which does not tend monotonically to zero as* $t \to \infty$.

PROOF OF THEOREM 1. Suppose $x(t)$ is a bounded nonoscillatory solution which does not tend monotonically to zero as $t \to \infty$. Assume without loss of generality, $x(t) > 0$. As in the proof of Theorem 1 of Ryder and Wend [3], by the boundedness of $x(t)$, we have the following:

(4) $$\lim_{t \to \infty} x^{(i)}(t) = 0 \qquad (i = 1, 2, \cdots, n-1),$$

(5) $$-x'(t) = \int_t^{\infty} \frac{(u-t)^{n-2}}{(n-2)!} f(u, x(u), x'(u), \cdots, x^{(n-2)}(u))\, du \geqq 0.$$

By (5), $x(t)$ decreases to a limit $L \geqq 0$ and by the assumptions on $x(t)$, $L > 0$. Integrating (5) from T to ∞, we have

(6) $$x(t) > x(t) - L \geqq \int_T^{\infty} \frac{(u-T)^{n-1}}{(n-1)!} f(u, x(u), x'(u), \cdots, x^{(n-2)}(u))\, du.$$

The generalized strong continuity of $f(t, x_1, x_2, \cdots, x_{n-1})$ implies that for $\varepsilon=\frac{1}{2}$ there exist $\delta>0$, $T\geqq 0$ and $L_\delta\in[L, L+\delta]$ such that for all x_i satisfying $|x_i-k_i|\leqq\delta$ $(i=2, \cdots, n-1)$, and for all $x_1\in[L, L+\delta]$ and $t\geqq T$,

$$f(t, x_1, x_2, \cdots, x_{n-1}) \geqq \tfrac{1}{2}f(t, L_\delta, k_2, \cdots, k_{n-1}).$$

From (4) it follows that there exists a $T_0\geqq T$ sufficiently large that for all $t\geqq T_0$, $x(t)$ satisfies $L\leqq x(t)\leqq L+\delta$ and

$$|x^{(i)}(t) - 0| \leqq \delta \qquad (i = 1, 2, \cdots, n-2).$$

Thus we obtain, for $t\geqq T_0$,

$$0 < \tfrac{1}{2}f(t, L_\delta, 0, \cdots, 0) < f(t, x(t), x'(t), \cdots, x^{(n-2)}(t)).$$

Accordingly, by (6), we have

$$x(t) \geqq \frac{1}{2}\int_T^\infty \frac{(u-T)^{n-1}}{(n-1)!} f(u, L_\delta, 0, \cdots, 0)\, du,$$

which implies (3).

Conversely, suppose that (3) holds for some constants $c\neq 0$ and k_i $(i=2, \cdots, n-1)$. Then we can prove there exists a solution $(x_0(t), \cdots, x_{n-2}(t))$ to the following system of integral equations:

$$\begin{aligned}
x_{n-2}(t) &= k_{n-1} - \int_t^\infty (s-t) f(s, x_0(s), \cdots, x_{n-2}(s))\, ds, \\
x_{n-3}(t) &= k_{n-2} + \int_t^\infty \frac{(s-t)^2}{2!} f(s, x_0(s), \cdots, x_{n-2}(s))\, ds, \qquad (7)\\
&\vdots \\
x_0(t) &= c + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, x_0(s), \cdots, x_{n-2}(s))\, ds.
\end{aligned}$$

Note that we can obtain

$$0 < c \leqq x_{0,N}(t) \leqq c + c/M \qquad (M \text{ is sufficiently large}),$$

by the same argument as in the even case (cf. [2], [4]).

Then, by using the method of successive approximations (cf. [2], [4]), we get a solution $(x_0(t), \cdots, x_{n-2}(t))$ and it is clear that $x_0(t)$ is a desired nonoscillatory solution of $(*)$ which does not tend monotonically to zero as $t\to\infty$.

3. Oscillation theorems.

THEOREM 2. *Assume that n is even and that condition* (1) *holds. Let* $f(t, x_1, x_2, \cdots, x_{n-1})$ *be generalized strongly continuous from the left for* $x_1>0$ *and generalized strongly continuous from the right for* $x_1<0$ *and suppose there exists a function* $\phi(x)$ *with the following properties*:

(a) $\phi(x)$ *is a nondecreasing continuous function of x satisfying* $x\phi(x)>0$ *whenever* $x\neq 0$;

(b) *there exists* $c\neq 0$ *and* k_i $(i=2, \cdots, n-1)$, *such that*

$$\text{(b1)} \qquad \liminf_{|x_1|\to\infty} \frac{f(t, x_1, x_2, \cdots, x_{n-1})}{\phi(x_1)} \geqq k\,|f(t, c, k_2, \cdots, k_{n-1})|$$

for some positive constant k and for all $t\geqq T$ *and*

$$\text{(b2)} \qquad \lim_{|x|\to\infty} \left|\int^{x} (du/\phi(u))\right| < \infty.$$

Then a necessary and sufficient condition for $(*)$ *to be oscillatory is that*

$$\text{(8)} \qquad \left|\int^{\infty} t^{n-1} f(t, c, k_2, \cdots, k_{n-1})\, dt\right| = \infty$$

for all constants c $(\neq 0)$ *and* k_i $(i=2, \cdots, n-1)$.

PROOF. Assume that (8) does not hold. Then (3) holds for some $c\neq 0$ and k_i $(i=2, \cdots, n-1)$. Hence by Theorem A, equation $(*)$ has a bounded nonoscillatory solution, so that condition (8) is necessary.

Conversely, let $x(t)>0$ be a nonoscillatory solution of $(*)$. In view of Ryder and Wend's [3] arguments, $x(t)$ must be nondecreasing and hence must tend to a limit, finite or infinite. Suppose first that the limit is finite, i.e. $\lim_{t\to\infty} x(t)=L$ (>0). Then, we obtain (cf. [1, Theorem 1]) for some L_δ and t sufficiently large

$$\int_t^{\infty} u^{n-1} f(u, L_\delta, 0, \cdots, 0)\, du < \infty$$

which contradicts (8).

Next, we turn to the case $\lim_{t\to\infty} x(t)=\infty$. By again using the argument of Ryder and Wend [3], we have for sufficiently large t, say $t\geqq T$,

$$\text{(9)} \qquad x'(t) \geqq \int_t^{\infty} \frac{(u-t)^{n-2}}{(n-2)!} f(u, x(u), x'(u), \cdots, x^{(n-2)}(u))\, du$$

if $x'(t), \cdots, x^{(n-1)}(t)$ tend monotonically to zero as $t \to \infty$, or

$$x'(t) > \int_t^\infty \frac{(t - t_1)^{n-2}}{(n-2)!} f(u, x(u), x'(u), \cdots, x^{(n-2)}(u))\, du, \tag{10}$$

where $t_1 < t$ is sufficiently large, if $x'(t)$ does not tend monotonically to zero as $t \to \infty$.

In case (9) holds, multiply each side of the inequality in (9) by $(\phi(x))^{-1}$. Since $\phi(x)$ is nondecreasing in x and $x(t)$ is nondecreasing in t, we obtain

$$\begin{aligned} \frac{x'(t)}{\phi(x(t))} &\geqq \frac{1}{\phi(x(t))} \int_t^\infty \frac{(u-t)^{n-2}}{(n-2)!} f(u, x(u), x'(u), \cdots, x^{(n-2)}(u))\, du \\ &\geqq \int_t^\infty \frac{f(u, x(u), x'(u), \cdots, x^{(n-2)}(u))}{\phi(x(u))} \frac{(u-t)^{n-2}}{(n-2)!}\, du. \end{aligned} \tag{11}$$

Integrating (11) from T to $t \geqq T$,

$$\begin{aligned} \int_{x(T)}^{x(t)} \frac{du}{\phi(u)} &\geqq \int_T^t \int_s^\infty \frac{f(u, x(u), x'(u), \cdots, x^{(n-2)}(u))}{\phi(x(u))} \frac{(n-s)^{n-2}}{(n-2)!}\, du\, d \\ &\geqq \int_T^t \frac{f(u, x(u), x'(u), \cdots, x^{(n-2)}(u))}{\phi(x(u))} \frac{(u-T)^{n-1}}{(n-1)!}\, du. \end{aligned} \tag{12}$$

By (b1) and the fact that $\lim_{t\to\infty} x(t) = \infty$, we may also choose T so large that, for $t \geqq T$,

$$\frac{f(t, x(t), x'(t), \cdots, x^{(n-2)}(t))}{\phi(x(t))} \geqq \frac{k}{2} |f(t, c, k_2, \cdots, k_{n-1})| > 0.$$

Substituting the above inequality in (12) and letting $t \to \infty$, we obtain a contradiction to (b2) if (8) holds.

In case (10) holds, an argument similar to that above again leads to a contradiction to (b2). This proves the sufficiency part of the theorem.

THEOREM 3. *In addition to the hypotheses* (a) *and* (b) *of Theorem* 2, *assume n is odd and that condition* (1) *holds. Let $f(t, x_1, \cdots, x_{n-1})$ be generalized strongly continuous. Then condition* (8) *is a necessary and sufficient condition for the solution of* (*) *to be oscillatory or tend monotonically to zero as $t \to \infty$.*

PROOF. The necessity follows from Theorem 1.

To prove the sufficiency, suppose $x(t) > 0$ is a nonoscillatory solution of (*) not tending monotonically to zero as $t \to \infty$.

In this case, we find that $x(t)$ satisfies either (5) or (10) by the argument used in Ryder and Wend [3]. If $x(t)$ satisfies (5), then our Theorem 3 follows from Theorem 1, and if $x(t)$ satisfies (10), then it follows as in the proof of Theorem 2.

REMARKS. For equation (2), Theorem 2 and Theorem 3 remain valid under the more severe condition that f be bounded for all t for each n-tuple $(x_1, x_2, \cdots, x_n)$ (cf. [2]). But we note here that the proofs of the sufficiency parts of Theorems A, 1, 2 and 3 remain valid for the more general equation (2). Only in the necessity parts does the boundedness of f in t, $-\infty<t<\infty$, enter in—in solving the system of integral equations as in [2]. Our results also extend Theorem 2 of Ryder and Wend [3].

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REFERENCES

1. H. Onose, *Oscillatory properties of solutions of even order differential equations*, Pacific J. Math. **38** (1971), 747–757.

2. ———, *Oscillatory property of ordinal differential equations of arbitrary order*, J. Differential Equations **7** (1970), 454–458. MR **41** #2116.

3. G. H. Ryder and D. V. V. Wend, *Oscillation of solutions of certain ordinary differential equations of nth order*, Proc. Amer. Math. Soc. **25** (1970), 463–469. MR **41** #5710.

4. J. S. W. Wong, *On second order nonlinear oscillation*, Funkcial. Ekvac. **11** (1968), 207–234, MR **39** #7221.

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