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ON THE RING OF QUOTIENTS OF A GROUP RING[1]

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ABSTRACT. In this paper the following extension of a result of Martha Smith is proved. THEOREM. *Let $K[G]$ be a group ring which is an order in a ring Q. Then the center of $K[G]$ is an order in the center of Q.*

Let $K[G]$ denote the group ring of G over the field K. The main result of this paper is

THEOREM. *Let $K[G]$ be a group ring which is an order in a ring Q. Then the center of $K[G]$ is an order in the center of Q.*

This was first proved by Martha Smith [**2**] in a number of special cases and later proved [**1**] for all semiprime group rings. We follow the notation of [**1**]. Thus, in particular, $\Delta(G)$ denotes the finite conjugate subgroup of G and $\theta:K[G]\to K[\Delta(G)]$ is the natural projection.

LEMMA 1. *Let $H\subseteq\Delta(G)$ be a finitely generated normal subgroup of G. Then*

(i) $[G:C_G(H)]<\infty$.

(ii) *H has a torsion free central subgroup Z of finite index which is normal in G.*

(iii) *The ring of quotients $K[Z]^{-1}K[H]$ is K-isomorphic to $F^t[H/Z]$, a twisted group ring of the finite group H/Z over the field $F=K[Z]^{-1}K[Z]$.*

PROOF. (i) Let $H=\langle h_1, h_2, \cdots, h_n\rangle$. Since $H\subseteq\Delta(G)$ we have $[G:C_G(h_i)]<\infty$. Hence $C_G(H)=\bigcap_1^n C_G(h_i)$ has finite index in G.

(ii) By (i) we see that $Z(H)=H\cap C_G(H)$ has finite index in H and certainly $Z(H)$ is normal in G. Since H is finitely generated and $[H:Z(H)]<\infty$, it follows that $Z(H)$ is a finitely generated abelian group. Thus $Z(H)=T\times A$ where T is a finitely generated torsion free abelian group and A is finite of order k. If $Z=\{x^k \mid x\in Z(H)\}$, then clearly Z is normal in G and Z is a torsion free central subgroup of H of finite index.

(iii) Now by Lemma 2.4 of [**1**] no nonzero element of $K[Z]$ is a zero divisor in $K[G]$ since Z is torsion free abelian. Since Z is central in $K[H]$,

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it is then trivial to form the ring of quotients $E=K[Z]^{-1}K[H]$. This is the set of all formal fractions $\eta^{-1}\alpha$ with $\eta\in K[Z]$, $\eta\neq 0$, $\alpha\in K[H]$ and with the usual identifications made. If $F=K[Z]^{-1}K[Z]$, then F is certainly a central subfield of E so E is an F-algebra. For each $x\in H/Z$ let $\bar{x}\in H$ be a coset representative. Then it is easy to see that $\{\bar{x}|x\in H/Z\}$ is an F-basis for the associative algebra E. Moreover for $x, y\in H/Z$ we have

$$\bar{x}\bar{y} = z\overline{xy} \quad \text{for some } z \in Z.$$

Since z is a nonzero element of F we conclude that $E\simeq F^t[H/Z]$.

LEMMA 2. *Let α be an element of $\boldsymbol{Z}(K[G])$, the center of the group ring. Then α is a zero divisor in $K[G]$ if and only if it is a zero divisor in $\boldsymbol{Z}(K[G])$.*

PROOF. If α is a zero divisor in $\boldsymbol{Z}(K[G])$ then it is certainly a zero divisor in $K[G]$. Assume now that α is a zero divisor in $K[G]$ and let $\gamma\in K[G]$, $\gamma\neq 0$ with $\alpha\gamma=0$. Let $H=\langle \operatorname{Supp}\alpha\rangle$. Since α is central, H is a finitely generated subgroup of $\Delta(G)$ which is normal in G. We use that notation and results of Lemma 1. Then $\alpha\in E=K[Z]^{-1}K[H]$ which is a finite dimensional algebra over F and hence α satisfies a polynomial over F. By rationalizing the denominators we can assume that $\alpha^n\beta=0$ where $\beta=\beta_0+\beta_1\alpha+\cdots+\beta_s\alpha^s$ with $\beta_i\in K[Z]$, $\beta_0\neq 0$.

Now the finite group $G/C_G(H)$ acts on $K[H]$ and let $x_1=1, x_2, \cdots, x_r$ be a full set of coset representatives for $C_G(H)$ in G. Set $\tilde{\beta}=\beta^{x_1}\beta^{x_2}\cdots\beta^{x_r}$. Since Z is normal in G and α is central in $K[G]$ we see that all β^{x_i} are polynomials in α with coefficients in $K[Z]$. Since Z is abelian this then implies that all β^{x_i} commute and hence clearly $\tilde{\beta}$ is central in $K[G]$. Now

$$\tilde{\beta} = (\beta_0^{x_1}\beta_0^{x_2}\cdots\beta_0^{x_r}) + \delta\alpha$$

for a suitable $\delta\in K[G]$ so we have

$$\tilde{\beta}\gamma = \beta_0^{x_1}\beta_0^{x_2}\cdots\beta_0^{x_r}\gamma.$$

Thus by Lemma 2.4 of [1], $\tilde{\beta}\gamma\neq 0$ since $\beta_0\neq 0$ implies that $\beta_0^{x_i}$ is not a zero divisor in $K[G]$. Hence $\tilde{\beta}\neq 0$.

Now $\beta^{x_1}=\beta$ so $\alpha^n\tilde{\beta}=0$. Since $\alpha^0\tilde{\beta}=\tilde{\beta}\neq 0$ we can choose $m\geqq 0$ maximal with $\alpha^m\tilde{\beta}\neq 0$. Then $\alpha(\alpha^m\tilde{\beta})=\alpha^{m+1}\tilde{\beta}=0$. Finally $\alpha^m\tilde{\beta}\in\boldsymbol{Z}(K[G])$ so α is a zero divisor in $\boldsymbol{Z}(K[G])$ and the result follows.

A ring R is said to be self-injective if R as a right R-module is injective. The following lemma is well known.

LEMMA 3. *Let R be a self-injective ring. Then every proper finitely generated left ideal has a proper right annihilator.*

PROOF. Let I be the left ideal $I=R\alpha_1+R\alpha_2+\cdots+R\alpha_n$ and assume that I has no proper right annihilator. Define

$$\sigma: R \to R \dotplus R \dotplus \cdots \dotplus R \quad (n \text{ times})$$

by

$$\sigma(r) = (\alpha_1 r, \alpha_2 r, \cdots, \alpha_n r)$$

for $r\in R$. This is clearly an R-homomorphism. If $\sigma(r)=0$ then $Ir=0$ so $r=0$. This σ is an injection.

Since R is self-injective, there exists a back map τ. Say

$$\tau(0, 0, \cdots, 1, \cdots, 0) = \beta_i \in R$$

where the 1 is in the ith position. Then

$$1 = \tau\sigma(1) = \tau(\alpha_1, \alpha_2, \cdots, \alpha_n) = \beta_1\alpha_1 + \beta_2\alpha_2 + \cdots + \beta_n\alpha_n \in I$$

and $I=R$.

The following is known for ordinary group rings.

LEMMA 4. *Let $F^t[W]$ be a twisted group ring of the finite group W over the field F. Then $F^t[W]$ is self-injective.*

PROOF. Since $\bar{1}$ is a scalar multiple of $1\in F^t[W]$ we may assume without loss of generality that $\bar{1}=1$. Let $\tau: F^t[W]\to F$ be the trace map. That is, if $\alpha=\sum a_x\bar{x}$ then $\tau(\alpha)=a_1$. Since $a_x=\tau(\alpha\bar{x}^{-1})$, we have $\alpha=\sum_{x\in W}\tau(\alpha\bar{x}^{-1})\bar{x}$.

Let $U\subseteq V$ be $F^t[W]$-modules and let $\sigma: U\to F^t[W]$ be a given $F^t[W]$-homomorphism. Then $\tau\sigma: U\to F$ and since F is a field we can extend $\tau\sigma$ to a map $\varphi: V\to F$ which is F-linear.

We now define $\tilde{\varphi}: V\to F^t[W]$ by

$$\tilde{\varphi}(v) = \sum_{x\in W} \varphi(v\bar{x}^{-1})\bar{x}.$$

Then certainly $\tilde{\varphi}$ is an F-linear map. Suppose $x, g\in W$. Then

$$\bar{x}\bar{g} = a\overline{xg} \quad \text{for some } a \in F,\ a \neq 0$$

so we have

$$\bar{g}^{-1}\bar{x}^{-1} = (\bar{x}\bar{g})^{-1} = (a\overline{xg})^{-1} = \overline{xg}^{-1}a^{-1}$$

and since φ is F-linear

$$\varphi(v\bar{g}^{-1}\bar{x}^{-1})\bar{x}\bar{g} = \varphi(v\overline{xg}^{-1}a^{-1})a\overline{xg} = \varphi(v\overline{xg}^{-1})\overline{xg}.$$

Now let $g\in W$. Then by the above

$$\tilde{\varphi}(v\bar{g}^{-1}) = \left(\sum_{x\in W}\varphi(v\bar{g}^{-1}\bar{x}^{-1})\bar{x}\bar{g}\right)\bar{g}^{-1} = \left(\sum_{x\in W}\varphi(v\overline{xg}^{-1})\overline{xg}\right)\bar{g}^{-1} = \tilde{\varphi}(v)\bar{g}^{-1}.$$

Thus $\tilde{\varphi}$ is an $F^t[W]$-homomorphism.

Finally for $u\in U$,

$$\tilde{\varphi}(u) = \sum_{x\in W} \varphi(u\bar{x}^{-1})\bar{x} = \sum_{x\in W} \tau(\sigma(u\bar{x}^{-1}))\bar{x}$$
$$= \sum_{x\in W} \tau(\sigma(u)\bar{x}^{-1})\bar{x} = \sigma(u)$$

since σ is an $F^t[W]$-homomorphism. Thus $\tilde{\varphi}$ extends σ and $F^t[W]$ is injective.

LEMMA 5. *Let $\alpha\in K[G]$. If α is not a left divisor of zero, then there exists $\gamma\in K[G]$ such that $\theta(\gamma\alpha)$ is central and is not a zero divisor in $K[G]$.*

PROOF. Write $\alpha=\sum_1^n x_i\alpha_i$ with $\alpha_i\in K[\Delta]$ and with the x_i in distinct cosets of Δ. Let H be the subgroup of G generated by the elements in the support of all α_i and their finitely many conjugates. Then H is a finitely generated normal subgroup of G and $H\subseteq\Delta(G)$. We use the results and notation of Lemma 1.

Now $\alpha_i\in K[H]\subseteq E=K[Z]^{-1}K[H]$ so we can define $I=E\alpha_1+E\alpha_2+\cdots+E\alpha_n$ to be a finitely generated left ideal of E. Observe that $E\simeq F^t[H/Z]$; so by Lemma 4, E is self-injective. Thus by Lemma 3 either $I=E$ or I has a proper right annihilator.

Suppose I has a proper right annihilator element, say $\eta^{-1}\mathscr{E}\in E$ with $\eta\in K[Z]$, $\eta\neq 0$, $\mathscr{E}\in K[H]$, $\mathscr{E}\neq 0$. Then clearly $I\mathscr{E}=0$ so $\alpha_i\mathscr{E}=0$ for all i and hence $\alpha\mathscr{E}=0$. Since α is not a left divisor of zero this is a contradiction and we conclude that $I=E$.

Now $I=E$ so $1\in I$. If we write 1 as a sum of left E multiples of the α_i and then rationalize the denominators we see that there exists $\beta\in K[Z]$, $\beta\neq 0$, with $\beta\in K[H]\alpha_1+K[H]\alpha_2+\cdots+K[H]\alpha_n$. Now the finite group $G/C_G(H)$ acts on $K[Z]$ and let $y_1, y_2, \cdots, y_s=1$ be a full set of coset representatives for $C_G(H)$ in G. Set $\tilde{\beta}=\beta^{y_1}\beta^{y_2}\cdots\beta^{y_s}$. By Lemma 2.4 of [**1**] we see that $\tilde{\beta}\neq 0$ and in fact $\tilde{\beta}$ is not a zero divisor in $K[G]$. Moreover since Z is abelian, all β^{y_i} commute and hence $\tilde{\beta}$ is clearly central in $K[G]$.

Since $y_s=1$, $\tilde{\beta}\in\sum K[H]\alpha_i$ so we can write $\tilde{\beta}=\sum_1^n \gamma_i\alpha_i$ with $\gamma_i\in K[H]$. Set $\gamma=\sum_1^n \gamma_i x_i^{-1}$. Then $\gamma\alpha=\sum_{ij}\gamma_i x_i^{-1}x_j\alpha_j$. If $i\neq j$, then clearly $\mathrm{Supp}(\gamma_i x_i^{-1}x_j\alpha_j)$ is disjoint from $\Delta(G)$. Thus $\theta(\gamma\alpha)=\sum_1^n \gamma_i\alpha_i=\tilde{\beta}$ and the lemma is proved.

PROOF OF THE THEOREM. Clearly $Z(Q)\supseteq Z(K[G])$. Let $\alpha\in Z(K[G])$ be an element which is not a zero divisor in $Z(K[G])$. Then by Lemma 2, α is not a zero divisor in $K[G]$ so α is invertible in Q. Clearly $\alpha^{-1}\in Z(Q)$.

Now let $\rho\in Z(Q)$. Then $\rho=\alpha^{-1}\beta$ where α, $\beta\in K[G]$ and α is not a zero divisor in $K[G]$. Thus for all $\omega\in Q$ we have $\omega\alpha^{-1}\beta=\alpha^{-1}\beta\omega$ so $\alpha\omega\alpha^{-1}\beta=\beta\omega$. Set $\omega=x\alpha$. Then for all $x\in G$, $\alpha x\beta=\beta x\alpha$. Now by Lemma 5 there exists $\gamma\in K[G]$ such that $\theta(\gamma\alpha)\in Z(K[G])$ is not a zero divisor in $K[G]$. Multiplying the above equation on the left by γ then yields $(\gamma\alpha)x\beta=(\gamma\beta)x\alpha$ for all $x\in G$ and by Lemma 1.3 of [1] we have $\theta(\gamma\alpha)\beta=\theta(\gamma\beta)\alpha$.

Set $\mathscr{E}=\theta(\gamma\alpha)$, $\eta=\theta(\gamma\beta)$. Since $\mathscr{E}\in Z(K[G])$ is not a zero divisor in $K[G]$ we have $\mathscr{E}^{-1}\eta\in Q$ and since $\alpha^{-1}\beta\in Z(Q)$ we obtain, from $\mathscr{E}\beta=\eta\alpha$,

$$\mathscr{E}^{-1}\eta = \beta\alpha^{-1} = \alpha(\alpha^{-1}\beta)\alpha^{-1} = \alpha^{-1}\beta = \rho.$$

Finally $\mathscr{E}$, $\rho\in Z(Q)$ so $\eta=\mathscr{E}\rho\in Z(Q)\cap K[G]=Z(K[G])$ and $\rho=\mathscr{E}^{-1}\eta$ is a quotient of elements in $Z(K[G])$. The result follows.

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