

THE SCHWARZIAN DERIVATIVE AND UNIVALENT FUNCTIONS

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ABSTRACT. In this paper we prove under certain conditions the function $w=f(z)$ is univalent in $|z| < 1$.

It is customary to formulate the inequalities of the "Verzerrungssatz" type for analytic functions $w=f(z)$, schlicht in the unit circle, with reference to a specific normalization. The two normalizations mainly used are: (a) $f(z)$ is finite in $|z| < 1$, $f(0)=0$, $f'(0)=1$; (b) $f(z)$ has a pole at $z=0$ with the residue 1. If we want to obtain inequalities which are independent of any particular normalization, we have to use quantities which are invariant with regard to an arbitrary linear transformation of the z -plane. The simplest quantity of this type is the Schwarzian differential parameter

$$\{w, z\} = (w''/w')' - \frac{1}{2}(w''/w')^2,$$

also called the Schwarzian derivative of w with regard to z .

It is easy to obtain an upper bound for $\{w, z\}$ by a simple transformation of the classical inequality $|a_1| \leq 1$ valid for functions $w=f(z)=z^{-1}+a_0+a_1z+\dots$ schlicht in the unit circle. Indeed, applying this inequality to the coefficient of z in the expansion of the schlicht function

$$\begin{aligned} g(z) &= \frac{f'(x)(1 - |x|^2)}{f((z+x)/(1+\bar{x}z)) - f(x)} \\ &= \frac{1}{z} + \bar{x} - \frac{1}{2} \frac{f''(x)}{f'(x)} (1 - |x|^2) \\ &\quad - \frac{1}{6} (1 - |x|^2)^2 \left[\left(\frac{f''(x)}{f'(x)} \right)' - \frac{1}{2} \left(\frac{f''(x)}{f'(x)} \right)^2 \right] z + \dots, \quad |x| < 1, \end{aligned}$$

we obtain $|\{w, z\}| \leq 6/(1 - |z|^2)^2$ [3, p. 226].

It is known that by replacing the number 6 in this inequality by 2, this necessary condition for the univalence of $f(z)$ in $|z| < 1$ becomes sufficient ([4], [2], [5], [1]).

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Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular in $|z| < 1$ and

$$g(z) = \frac{f'(x)(1 - |x|^2)}{f((z+x)/(1+\bar{x}z)) - f(x)} = \frac{1}{z} + h(z, x), \quad |x| < 1.$$

Then $f(z)$ is univalent in $|z| < 1$ if and only if $g(z)$ is univalent in $|z| < 1$. In this case, the above theorem can be stated as follows:

THEOREM (NEHARI [1]). *In order that the function $w=f(z)$ be univalent in $|z| < 1$, it is necessary that*

$$|h'(0, x)| = \frac{1}{6}(1 - |x|^2)^2 \{f, x\} \leq 1 \quad \text{for } |x| < 1$$

and sufficient that

$$|h'(0, x)| \leq \frac{1}{3} \quad \text{for } |x| < 1.$$

We can now prove the following similar theorem for a sufficient condition that the function $w=f(z)$ be univalent in $|z| < 1$.

THEOREM 1. *In order that the function $w=f(z)$ be univalent in $|z| < 1$, it is sufficient that*

$$|h'(z, x)| \leq 1 \quad \text{for } |z| < 1$$

where x is in the unit disk.

PROOF. Evidently, $g(z) - 1/z = h(z, x)$ is regular in $|z| < 1$ and

$$(g(z_2) - 1/z_2) - (g(z_1) - 1/z_1) = h(z_2, x) - h(z_1, x) = \int_{z_1}^{z_2} h'(z, x) dz$$

where the integral is taken on the line segment $z_1 z_2$, $z_1 \neq z_2$, $|z_1| < 1$ and $|z_2| < 1$. Putting

$$z = z_1 + t(z_2 - z_1), \quad 0 \leq t \leq 1,$$

and

$$dz = (z_2 - z_1) dt$$

the above integral can be written in the form

$$\int_0^1 (z_2 - z_1) h'(z, x) dt.$$

Multiplying $|-z_1 z_2 / (z_2 - z_1)|$ on both sides we have

$$\begin{aligned} \left| \frac{(g(z_1) - 1/z_1)z_1 z_2 - (g(z_2) - 1/z_2)z_1 z_2}{z_2 - z_1} - 1 \right| &= \left| z_1 z_2 \int_0^1 h'(z, x) dt \right| \\ &< \int_0^1 |h'(z, x)| dt \leq \int_0^1 dt = 1. \end{aligned}$$

This shows that $g(z_2) - g(z_1) \neq 0$ for $z_1 \neq z_2$ and therefore $f(z)$ is univalent in $|z| < 1$.

THEOREM 2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular in $|z| < 1$ and*

$$\operatorname{Re}(f^2(z)/z^2 f'(z)) \geq \frac{1}{2} \quad \text{in } |z| < 1.$$

Then $f(z)$ is univalent in $|z| < 1$.

PROOF. Applying Theorem 1 to $f(z)$, $f(z)$ is univalent in $|z| < 1$ if $|h'(z, 0)| \leq 1$ in $|z| < 1$. By the maximum principle, $|z^2 h'(z, 0)| \leq 1$, in $|z| < 1$, implies $|h'(z, 0)| \leq 1$ in $|z| < 1$. On the other hand, it is easily confirmed that

$$|-z^2 h'(z, 0)| = |z^2 f'(z)/f^2(z) - 1|.$$

Therefore if we suppose that

$$|z^2 f'(z)/f^2(z) - 1| \leq 1 \quad \text{in } |z| < 1$$

or

$$\operatorname{Re}(f^2(z)/z^2 f'(z)) \geq \frac{1}{2} \quad \text{in } |z| < 1,$$

then $f(z)$ is univalent in $|z| < 1$.

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