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AN IDEAL CRITERION FOR TORSION FREENESS

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ABSTRACT. Auslander and Bridger have shown that, under conditions somewhat weaker than finite projective dimension, the "torsion freeness" properties of a module M (e.g. being reflexive, being the kth syzygy of another module) are determined by certain arithmetic conditions on the $\mathrm{Ext}^i(M, R)$. In this paper it is shown that a single ideal, the intersection of the annihilators of these modules, gives this same information. This ideal is then related to the Fitting invariants and invariant factors of M, and a computation is made of certain syzygies of a quotient of M (by a regular M-sequence).

Introduction. In [2] it is shown that under certain reasonable hypotheses (somewhat weaker than having finite projective dimension), the torsion freeness properties of a module M are determined by the grades of the modules $\mathrm{Ext}^i(M, R)$ for $i>0$. These numbers, in turn, depend only on the ideals $\mathfrak{A}_i=\mathrm{rad}[\mathrm{Ann}\,\mathrm{Ext}^i(M, R)]$. Denoting by $\gamma(M)$ the intersection of these ideals, the author has shown that $\gamma(M)$ likewise determines the torsion freeness properties of M. In this paper this result will be partially generalized, and an explicit calculation made exhibiting the torsion freeness properties (§3). In §2 we relate $\gamma(M)$ to the Fitting invariants and invariant factors of M.

Unless otherwise specified, all rings will be commutative and noetherian and modules finitely generated (hence finitely presented).

1. **The ideals $\beta(M)$ and $\gamma(M)$.** Although M is projective if and only if $\mathrm{Ext}^1(M, -)=0$, one need not test $\mathrm{Ext}^1(M, -)$ on all modules. Let us write $\Omega M=\Omega^1 M=\ker(P\to M)$ where $P\to M$ is any map of a projective module P onto M. If we agree to call two objects A and B *projectively* equivalent when $A+P\approx B+Q$ for projective objects P and Q, then it is well known that the projective "equivalence class" of ΩM depends only on that of M (see Lemma 10 below).

LEMMA 1. *M is projective if and only if* $\mathrm{Ext}^1(M, \Omega M)=0$.

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PROOF. From left to right the implication is clear; on the other hand, $\mathrm{Ext}^1(M, \Omega M)=0$ implies that the exact sequence $0 \to \Omega M \to P \to M \to 0$ splits, so M (and ΩM) are projective.

For future use, we define the ith *syzygy* of M, $\Omega^i M$, recursively by: $\Omega^i M = \Omega(\Omega^{i-1} M)$; $\Omega^\circ M = M$. While any two ith syzygies of M are projectively equivalent. it is unknown to this author whether a module projectively equivalent to an ith syzygy of M is, in general, actually realizable as such.

PROPOSITION 2. *Let $x \in R$, $x \neq 0$. The following statements are equivalent*:

(i) *$M_{\{x\}}$ is $R_{\{x\}}$-projective (here $M_{\{x\}}$ is the module of fractions with respect to powers of x).*

(ii) *$M \to M$ given by multiplication by x^i factors through a projective for i sufficiently large.*

(iii) *$x^i\,\mathrm{Ext}^n(M, -)=0$ for $n>0$ and i sufficiently large.*

(iv) *$x^i\,\mathrm{Ext}^1(M, \Omega M)=0$ for i sufficiently large.*

PROOF. Suppose (i) holds; let $F \xrightarrow{\eta} M \to 0$ be exact with F projective. The epimorphism $[\eta/1]: F_{\{x\}} \to M_{\{x\}}$ has a splitting which, since M is finitely presented, is of the form $[\alpha/x^t]$ where $\alpha: M \to F$. Thus,

$$[\eta/1][\alpha/x^t] = \text{identity of } M_{\{x\}}.$$

We conclude that $x^j(\eta \circ \alpha) = x^j x^t$ for some $j \geqq 0$. Thus, for $i=j+t$ we have the following commutative diagram:

$$\begin{tikzcd} M \arrow[rr, "x^i"] \arrow[dr, "\alpha"'] & & M \\ & F \arrow[ur, "x^j \eta"'] & \end{tikzcd}$$

which verifies (ii).

Now suppose that (ii) holds; then the diagram

$$\begin{tikzcd} M \arrow[rr, "x^i"] \arrow[dr] & & M \\ & F \arrow[ur] & \end{tikzcd}$$

induces

$$\begin{tikzcd} \mathrm{Ext}^1(M, -) & & \mathrm{Ext}^1(M, -) \arrow[ll] \arrow[dl] \\ & \mathrm{Ext}^1(F, -) = 0 \arrow[ul] & \end{tikzcd}$$

giving (iii).

The implication (iii)$\Rightarrow$(iv) is clear.

Finally, if (iv) holds then

$$0 = (\mathrm{Ext}^1_R(M, \Omega M))_{\{x\}} = \mathrm{Ext}^1_{R_{\{x\}}}(M_{\{x\}}, \Omega_{\{x\}} M_{\{x\}})$$

(forming syzygies is easily seen to commute with taking modules of fractions). The lemma now applies, so (i) holds.

Following the lead of this proposition, we have:

DEFINITION 3. $\beta(m)=\{x\in R \mid M_{\{x\}}$ is $R_{\{x\}}$-projective$\}$ (note that $\beta(M)$ as here defined is the radical of the ideal $\beta(M)$ defined in [**1**]).

Now if $M\neq 0$ happens to have finite projective dimension, then pd $M=\sup\{i \mid \mathrm{Ext}^i(M, R)\neq 0\}$; in particular, M is projective if and only if $\mathrm{Ext}^i(M, R)=0$ for all $i>0$. If we put $\mathfrak{A}_i=\mathrm{rad}(\mathrm{Ann}\,\mathrm{Ext}^i(M, R))$ and $\gamma(M)=\bigcap_{i>0}\mathfrak{A}_i$, then we have:

PROPOSITION 4. *If* pd $M<\infty$ *then* $\beta(M)=\gamma(M)$.

(This is an easy consequence of the above remarks and the fact that, for modules having finitely generated projective resolutions, taking Ext commutes with taking modules of fractions.)

2. Relation to invariant factors and Fitting invariants.

DEFINITION 5. Given a presentation $R^m \xrightarrow{\Delta} R^n \to M \to 0$, the pth *Fitting invariant* $F_p(M)$ is the ideal generated by the $(n-p+1)\times(n-p+1)$ minors of the matrix of Δ. (This is shown to depend only on M.)

DEFINITION 6. $\alpha_p(M)=\mathrm{Ann}(\bigwedge^p M)$ is called the pth *invariant factor* of M.

The following fact may be found in [**3**].

PROPOSITION 7. *For each* p, $F_p(M)$ *and* $\alpha_p(M)$ *have the same radicals.*

LEMMA 8. *Let* $\bar{\alpha}(M)=\bigcap_{\alpha_i(M)\neq 0}\alpha_i(M)$; *then* $\bar{\alpha}(M)\subset\beta(M)$.

PROOF. Let x be in $\bar{\alpha}(M)$. It suffices to prove that $M_\wp$ is $R_\wp$ projective for each prime $\wp$ not containing x (since these correspond to the primes of $R_{\{x\}}$). Suppose then that $x\notin\wp$ and observe that over the ring $R_\wp$ we have: $\mathrm{Ann}(\bigwedge^p M_\wp)=\alpha_p(M)\otimes R_\wp$. This last ideal is zero if $\alpha_p(M)=0$ and all of $R_\wp$ if $\alpha_p(M)\neq 0$, since x is then in $\alpha_p(M)$. Thus, the invariant factors of $M_\wp$ over $R_\wp$ are either zero or the whole ring. However, in [**1**, Proposition 2.3] it is shown that if $n=$exterior rank $E=\sup\{i \mid \bigwedge^i E\neq 0\}$ then (over local rings) E is free if and only if $\alpha_n(E)=0$. Applying this to $M_\wp$ gives our results.

PROPOSITION 9. *If R is a domain or* pd $M<\infty$ *then* rad $\bar{\alpha}(M)=\beta(M)$.

PROOF. One inclusion is the above lemma. To prove $\beta(M)\subset\mathrm{rad}\,\bar{\alpha}(M)$ we use a slight modification of the proof given in [**1**] for the case R a domain. Suppose $\alpha_p(M)\neq 0$ and $x\colon M\to M$ factors through a projective F.

We may assume F is free since it is easily shown that a map $A \to B$ factors through a projective if and only if it factors through *any* projective mapping onto B. We obtain first a commutative diagram:

$$\begin{tikzcd} M \arrow[rr, "x"] \arrow[dr] & & M \\ & F \arrow[ur] & \end{tikzcd}$$

which yields another:

$$\begin{tikzcd} \bigwedge^p M \arrow[rr, "x^p"] \arrow[dr] & & \bigwedge^p M \\ & \bigwedge^p F \arrow[ur] & \end{tikzcd}$$

Since F is free, it suffices to prove that grade $\bigwedge^p M>0$, this being equivalent to $\mathrm{Hom}(\bigwedge^p M, \mathrm{Free})=0$. When R is a domain, the grade condition holds since $\alpha_p(M)\neq 0$ means that $\alpha_p(M)=\mathrm{Ann} \bigwedge^p M$ contains an R-regular element. Let us go then to the case pd $M<\infty$. The question of grade being a local one, we may suppose R to be local. Therefore M has a finite free resolution. For such a module it is known **[5]** that the Fitting invariants are either zero (for $p\leqq\chi(M)=$Euler characteristic of M) or contain a nonzero divisor ($p>\chi(M)$). Since $\alpha_p(M)$ and $F_p(M)$ have the same radicals, as was pointed out in Proposition 7 above, we see that $\alpha_p(M)$ contains an R-regular element; therefore grade $\bigwedge^p M>0$ and so multiplication by x^p is zero on $\bigwedge^p M$.

3. **A calculation of syzygies.** As was pointed out above, the syzygies of a module are determined only up to projective equivalence. One way of expressing this fact is by the following well-known criterion:

LEMMA 10. *A is projectively equivalent to B if and only if* $\mathrm{Ext}^1(A, -)$ *and* $\mathrm{Ext}^1(B, -)$ *are naturally isomorphic functors. In particular, A is projectively equivalent to $\Omega^n M$ if and only if* $\mathrm{Ext}^1(A, -)\approx \mathrm{Ext}^{n+1}(M, -)$.

We now come to the main result of this section.

THEOREM 11. *Let $(x_1, \cdots, x_k)$ be an M-regular sequence contained in $\beta(M)$. Then there is a positive integer n such that, if $y_i=x_i^n$, then*

$$\Omega^k(M/(y_1, \cdots, y_k)M) \sim \sum_{i=0}^{k} \binom{k}{i} \cdot \Omega^i M.$$

(*Here $\sim$ denotes projective equivalence and $t\cdot N$ the direct sum of t copies of N.*)

PROOF. By Proposition 2 there is, for each i, an integer $n(i)$ such that $x_i^{n(i)}: M \to M$ factors through a projective P. Let $n = \max n(i)$.

Note that multiplication by x_i^n on $\Omega^s M$ also factors through a projective (*viz.* $\Omega^s P$) for each $s > 0$.

We proceed by induction. When $k=0$ there is nothing to prove. Suppose then the theorem holds for an integer $k \geq 0$, and let $x_1, \cdots, x_{k+1}$ be an M-regular sequence contained in $\beta(M)$. Then we have

$$\Omega^k(M/(y_1, \cdots, y_k)M) \sim \sum_{i=0}^{k} \binom{k}{i} \cdot \Omega^i M = (\mathrm{Id}+\Omega)^k(M).$$

To simplify notation let $\bar{M} = M/(x_1^n, \cdots, x_k^n)M$, $x = x_{k+1}$. Consider the exact sequence

$$0 \longrightarrow \bar{M} \xrightarrow{x^n} \bar{M} \longrightarrow \bar{M}/x^n\bar{M} \longrightarrow 0.$$

Note that $\bar{M}/x^n\bar{M} = M/(x_1^n, \cdots, x_k^n, x_{k+1}^n)M$. Part of the long exact sequence for Ext gives:

$$\begin{aligned} \mathrm{Ext}^{k+1}(\bar{M}, -) \xrightarrow{x^n} \mathrm{Ext}^{k+1}(\bar{M}, -) &\longrightarrow \mathrm{Ext}^{k+2}(\bar{M}/x^n\bar{M}, -) \\ &\longrightarrow \mathrm{Ext}^{k+2}(\bar{M}, -) \xrightarrow{x^n} \mathrm{Ext}^{k+2}(\bar{M}, -). \end{aligned}$$

Now multiplication by x^n on $\Omega^k\bar{M}$ factors through a projective by the induction hypothesis. Since $\mathrm{Ext}^{k+i}(\bar{M}, -) \approx \mathrm{Ext}^i(\Omega^k\bar{M}, -)$ we see that $x^n \,\mathrm{Ext}^{k+i}(\bar{M}, -) = 0$ and we obtain the short exact sequence:

$$0 \to \mathrm{Ext}^1(\Omega^k\bar{M}, -) \to \mathrm{Ext}^1(\Omega^{k+1}(\bar{M}/x^n\bar{M}), -) \to \mathrm{Ext}^1(\Omega^{k+1}\bar{M}, -) \to 0.$$

A short exact sequence of Ext functors is, however, known to split (see **[4]**). Therefore

$$\mathrm{Ext}^1(\Omega^{k+1}(\bar{M}/x^n\bar{M}), -) \approx \mathrm{Ext}^1(\Omega^k\bar{M} + \Omega^{k+1}\bar{M}, -).$$

By Lemma 10 above $\Omega^{k+1}(\bar{M}/x^n\bar{M}) \sim \Omega^k\bar{M} + \Omega^{k+1}\bar{M} = (\mathrm{Id}+\Omega)(\Omega^k\bar{M}) = (\mathrm{Id}+\Omega)(\mathrm{Id}+\Omega)^k(M) = \sum_{i=0}^{k+1} \binom{k+1}{i} \Omega^i M$.

This completes the induction and the proof.

4. **Torsion freeness.** In this section we find a (partial) converse to Theorem 1. Let us recall first the notion of k-torsion freeness; for further details see [2]. For a module M, choose a presentation $P_1 \to P_0 \to M \to 0$ and write $DM = \mathrm{coker}(P_0^* \to P_1^*)$ where $P^* = \mathrm{Hom}(P, R)$. Then DM depends (up to projective equivalence) only on M.

DEFINITION 12. M is k-torsion free if $\mathrm{Ext}^i(DM, R) = 0$ for $1 \leq i \leq k$.

The reason for this definition is the following exact sequence:

$$0 \to \mathrm{Ext}^1(DM, R) \to M \to M^{**} \to \mathrm{Ext}^2(DM, R) \to 0$$

so 1-torsion freeness is equivalent to torsion freeness, while 2-torsion freeness is equivalent to reflexivity.

REMARK. If N is a direct summand of M and M is k-torsion free, then so is N. In particular, if N is projectively equivalent to a k-t.f. module, N is itself k-t.f.

The proof of the following may be found in [2]:

PROPOSITION 13. *Consider the properties*:

(a) *M is k-torsion free.*

(a′) *There is an exact sequence* $0 \to M \to P_{k-1} \to \cdots \to P_0$ *where the* P_i *are projective and* $P_0^* \to \cdots \to P_{k-1}^* \to M^* \to 0$ *is exact.*

(b) *There is an exact sequence* $0 \to M \to P_{k-1} \to \cdots \to P_0$ *where the* P_i *are projective.*

(c) *M is projectively equivalent to $\Omega^k M$ for some N.*

(d) *Every R-regular sequence of length $\leqq k$ is M-regular.*

Then the following implications hold: (a)$\Leftrightarrow$(a′)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d). *If* $\operatorname{pd} M < \infty$ *then each of these is equivalent to*

(e) *grade* $\operatorname{Ext}^i(M, R) \geqq i+k$ *for each* $i > 0$.

REMARK. The hypothesis $\operatorname{pd} M > \infty$ may actually be replaced by the weaker condition $G(R_\wp) - \dim M_\wp < \infty$ for each prime $\wp$; details may be found in [2]. We now use the equivalence of these properties ($\operatorname{pd} M < \infty$) to prove

THEOREM 14. *Suppose* $\operatorname{pd} M < \infty$. *Then M is a k-syzygy (equivalently: M is k-torsion free) if and only if $\gamma(M)$ contains an M-regular sequence of length k.*

PROOF. Suppose $\gamma(M)$ contains an M-regular sequence of length k. Since $\gamma(M) = \beta(M)$ in this case—Proposition 4—we see that $M = \Omega^\circ M$ is a direct summand of $\Omega^k(M/(y_1, \cdots, y_k)M)$ by Theorem 11. By Proposition 13 this last module is k-torsion free. Finally, by the first remark of this section, M is k-t.f., being a direct summand of a k-t.f. module.

Conversely, suppose M is k-t.f. Then grade $\operatorname{Ext}^i(M, R) \geqq i+k$ for each $i > 0$. Let $\wp$ be any prime containing $\gamma(M) = \bigcap_{i>0} \operatorname{rad}[\operatorname{Ann} \operatorname{Ext}^i(M, R)]$. Then $\wp$ contains $\operatorname{Ann} \operatorname{Ext}^i(M, R)$ for some i. We conclude that $\wp$ contains an R-regular sequence of length (at least) k by the grade condition on $\operatorname{Ext}^i(M, R)$. Since M is k-torsion free, Proposition 13 shows that such a sequence is M-regular as well; $\wp$ being arbitrary, $\wp \supset \gamma(M)$, we conclude that $\gamma(M)$ contains an M-regular sequence of length k. This completes the proof.

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