

FIXED POINTS BY MEAN VALUE ITERATIONS

GORDON G. JOHNSON

ABSTRACT. If E is a convex compact subset of a Hilbert space, T is a strictly pseudocontractive function from E into E and x_1 is a point in E , then the point sequence $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of T , where for each positive integer n ,

$$x_{n+1} = [1/(n + 1)][Tx_n + nx_n].$$

In this paper it is shown that a technique of W. R. Mann [1] is fruitful in finding a fixed point of a function even though a "Picard" iteration may fail. The technique of Mann is similar to some of the techniques used in [2].

THEOREM. *If E is a compact convex subset of a Hilbert space H , T is a strictly pseudocontractive function from E into E and x_1 is any point in E , then the point sequence $\{x_i\}_{i=1}^{\infty}$ converges in the norm of H to a fixed point of T , where for each positive integer n , $x_{n+1} = [1/(n + 1)][Tx_n + nx_n]$.*

PROOF. By a well-known result of Schauder [3], T has a fixed point in E . Let p denote one such fixed point of T and x_1 a point in E . Since H is a Hilbert space, it is easily shown that for each positive integer n ,

$$\begin{aligned} \|x_{n+1} - p\|^2 &= [1/(n + 1)] \|Tx_n - p\|^2 \\ &\quad + [n/(n + 1)] \|x_n - p\|^2 - [n/(n + 1)^2] \|Tx_n - x_n\|^2 \end{aligned}$$

where $x_{n+1} = [1/(n + 1)][Tx_n + nx_n]$.

Since T is strictly pseudocontractive, there is a number k , $0 \leq k < 1$, such that if each of x and y is in E then

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k \|(I - T)x - (I - T)y\|^2.$$

Hence

$$\begin{aligned} \|Tx_n - p\|^2 &\leq \|x_n - p\|^2 + k \|x_n - Tx_n\|^2, \\ \|x_{n+1} - p\|^2 &\leq \|x_n - p\|^2 - [(n - k(n + 1))/(n + 1)^2] \|Tx_n - x_n\|^2. \end{aligned}$$

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There is a positive integer N such that $N/(N+1) > k$ and therefore if j is a positive integer

$$\|x_{N+j} - p\|^2 \leq \|x_{N+j-1} - p\|^2 - [(1-k)(N+j-1) - k]/(N+j)^2 \|Tx_{N+j-1} - x_{N+j-1}\|^2$$

from which it follows that

$$\|x_{N+j+1} - p\|^2 \leq \|x_N - p\|^2 - \sum_{t=0}^j [(1-k)(N+t) - k]/(N+t+1)^2 \|Tx_{N+t} - x_{N+t}\|^2.$$

Since E is bounded the above implies that

$$\sum_{t=0}^{\infty} [(1-k)(N+t) - k]/(N+t+1)^2 \|Tx_{N+t} - x_{N+t}\|^2 < \infty$$

and $\|x_{N+j+1} - p\| \leq \|x_{N+j} - p\|$ for $j=1, 2, \dots$.

Hence

$$\lim_{t \rightarrow \infty} \|Tx_{N+t} - x_{N+t}\| = 0.$$

Since E is compact there is a subsequence $\{x_{N+n_i}\}_{i=1}^{\infty}$ which converges to a point q in E . Notice that q is a fixed point of T .

Since q is a fixed point of T we have for each positive integer j that

$$\|x_{N+j+1} - q\| \leq \|x_{N+j} - q\|.$$

This last comment combined with the fact that $\{x_{N+n_i}\}_{i=1}^{\infty}$ converges to q yields the desired result, namely that $\{x_i\}_{i=1}^{\infty}$ converges to q , which is a fixed point of T .

REFERENCES

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TEXAS 77004