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NUMERICAL RANGE FOR CERTAIN CLASSES OF OPERATORS

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ABSTRACT. We prove a basic result which relates the structure of the spectrum to the interior of the numerical range. Using this result we derive corollaries concerning compact operators, quasi-nilpotents, and finite dimensional operators. In particular, we characterize finite dimensional convexoid operators.

1. **Introduction.** In this paper "operator" will mean a bounded linear transformation of the complex Hilbert space H into itself. The thrust of the conclusions that we obtain here is to show that the numerical range of an operator T can be described provided that $T-zI$ has closed range. For a general discussion of numerical range see Chapter 17 of [**4**].

2. **Preliminaries.** For an isolated eigenvalue z there are two different notions of eigenspace; the geometric eigenspace is just the kernel of $T-zI$ (which we simply write as $T-z$). The algebraic eigenspace associated with z is the range of an idempotent P defined by a contour integral according to the Banach space operational calculus (see pp. 178–181 of [**5**], for example). If the underlying Hilbert space is H then both PH and $(I-P)H$ are invariant under T, and the restriction of $T-z$ to PH (which we denote $T-z/PH$) is quasi-nilpotent. We shall say that an eigenvalue is a normal eigenvalue if the corresponding geometric eigenspace reduces T; if a normal eigenvalue is an isolated eigenvalue and the geometric multiplicity agrees with the algebraic multiplicity then we say that it is a normal-isolated eigenvalue. Clearly an isolated eigenvalue for a normal operator is a normal-isolated eigenvalue.

It will be convenient to denote by $W(T)$ the numerical range of T, i.e. $\{\langle Tf,f\rangle : \|f\|=1\}$, and the closure of the numerical range is denoted by $W(T)^-$. Finally, an operator is convexoid provided that $W(T)^-$ is the convex hull of the spectrum of T, denoted conv $\sigma(T)$. Note that it follows from Theorem 1.24 on p. 16 of [**8**] that every point of conv $\sigma(T)$ can be

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written as a convex linear combination of some four points of $\sigma(T)$. From this and the fact that the continuous image of a compact set is compact, it follows that conv $\sigma(T)$ is compact. Thus conv $\sigma(T)$ is closed.

3. Main results.

LEMMA 1. *If z is an eigenvalue of T and if z belongs to the topological boundary of $W(T)$, denoted* Bdry $W(T)$, *then* $\ker(T-z)=\ker(T^*-z)$.

PROOF. By choosing an appropriate θ and replacing T with $e^{i\theta}(T-\bar{z})$ we may assume that $z=0$ and Re $T\geqq 0$. Then for $f\in\ker T$ we have

$$0 = \operatorname{Re}\langle Tf,f\rangle = \langle(\operatorname{Re} T)f,f\rangle = \|(\operatorname{Re} T)^{1/2}f\|^2$$

and so $(\operatorname{Re} T)f=0$. It follows that $f\in\ker T^*$.

THEOREM 1. *For any z in the spectrum of T, i.e. $z\in\sigma(T)$, one of the following holds*: (1) *z belongs to the interior of the numerical range, i.e.* $z\in\operatorname{Int} W(T)$, (2) *$z$ is a normal-isolated eigenvalue*, (3) *the range of $T-z$ is not closed.*

PROOF. Suppose that $z=0$ and assume that both (1) and (3) above fail. Since T is not invertible either $\ker T\neq\{0\}$ or $\ker T^*\neq\{0\}$. In either case zero belongs to Bdry $W(T)$ and the lemma implies that $\ker T=\ker T^*\neq\{0\}$. Thus $\ker T$ reduces T and we note that $(\ker T)^\perp=(\ker T^*)^\perp=TH$. Thus $T=0\oplus T_1$ on $\ker T\oplus TH$ and since T_1 is one-to-one and onto, it is invertible. It follows that $T-z$ is invertible for all nonzero z with $|z|$ sufficiently small. Hence zero is a normal-isolated eigenvalue for T.

REMARK 1. The above theorem can also be deduced from Theorem 1 of Crabb in [**3**] using the theory of ascent and descent. For an account of the different definitions of numerical range in a Banach space see [**1**].

COROLLARY 1. *If T is compact and if $z\neq 0$ belongs to $\sigma(T)\cap$*Bdry $W(T)$ *then z is a normal-isolated eigenvalue.*

PROOF. If T is compact and $z\neq 0$ then $T-z$ has closed range.

For a discussion relevant to the next corollary see Problem 170 of [**14**].

COROLLARY 2. *If N is a nonzero quasi-nilpotent operator such that NH is closed then* $0\in\operatorname{Int} W(N)$.

PROOF. Since $0\in\sigma(N)$, we can apply Theorem 1 and conclude that either (1) or (2) holds. Since $\sigma(N)=\{0\}$, it is clear that the algebraic eigenspace associated with zero is the whole Hilbert space. So if (2) holds then N is the zero operator.

Consult Lemma 5 of [**2**] for a list of conditions sufficient for an operator to be convexoid.

THEOREM 2. *If T is convexoid and z is an extreme point of $W(T)^-$ then either z is a normal-isolated eigenvalue or else $(T-z)H$ is not closed.*

PROOF. It is clear that the extreme points of $W(T)^-$ are contained in Bdry $W(T)$ and so this follows from Theorem 1.

In **[6]** Moyls and Marcus obtain conditions which are necessary and sufficient for an operator on a finite dimensional Hilbert space to be convexoid. The next result is somewhat simpler than their theorem.

COROLLARY 3. *An operator T on a finite dimensional Hilbert space H is convexoid if and only if every extreme point of $W(T)$ is a normal-isolated eigenvalue.*

PROOF. Half of the corollary is immediate from Theorem 2. We assume that every extreme point of $W(T)$ is a normal-isolated eigenvalue and we denote this set of extreme points by $E(T)$. By the Hausdorff-Toeplitz theorem we know that $W(T)$ is a compact convex set and by the Krein-Milman theorem we know that conv $E(T)=W(T)$. Since $E(T)\subset\sigma(T)$, we have $W(T)\subset$ conv $\sigma(T)$. It is well known that $W(T)^-\supset\sigma(T)$ and since $W(T)$ is closed and convex, we have $W(T)\supset$ conv $\sigma(T)$. Thus the corollary is proved.

The next corollary was also noted by Moyls and Marcus.

COROLLARY 4. *If T is a convexoid operator on H and the dimension of H is not greater than four then T is normal.*

PROOF. By Corollary 3, $W(T)$ has at most four extreme points. If $W(T)$ has four extreme points then T is clearly normal. If $W(T)$ has three extreme points then T has a three dimensional reducing subspace such that T restricted to it is normal. Hence T is the direct sum of a normal operator and an operator on a one dimensional subspace and therefore it is normal. If $W(T)$ does not have more than two extreme points then either $W(T)$ is a line segment or it is one point. In the former case T is a linear function of a selfadjoint operator and in the latter case T is a scalar multiple of the identity. Hence T is normal.

REMARK 2. We give an example due to J. P. Williams which shows that the above corollary cannot be extended to the case where dim H is five. If the invertible convexoid operator T is normal then T^{-1} is normal and it follows that T^{-1} is convexoid. The operator that we construct is invertible and convexoid but its inverse is not convexoid. Let T_0 be the matrix operator $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ and recall that the numerical range of T_0 is the closed disc centered at zero with radius $\frac{1}{2}$. Thus the numerical range of $T_1=I+T_0$ is the closed disc centered at 1 with a radius of $\frac{1}{2}$. Let $\{z_2, z_3, z_4\}$ be points so placed that they are the vertices of an equilateral triangle which circumscribes the above disc. If $T_k=z_kI$ for $k=2, 3, 4$ and if $T=T_1\oplus T_2\oplus T_3\oplus T_4$

then T is invertible and convexoid but T^{-1} is not convexoid. (The inverse of T_1 is $I-T_0$.)

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