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RECOGNIZING MANIFOLDS AMONG GENERALIZED MANIFOLDS

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ABSTRACT. This paper provides various conditions, on the complement of a point in a generalized manifold M, which imply that M is a classical topological manifold. Similar characterizations are given for m-spheres and 3-cells.

This paper announces a few results in the classic quest for a property which characterizes the topological manifolds among the generalized manifolds.

It is well known, for example, that if a 3-gm M is a product space then it is a manifold. In [**1**], Raymond showed that the factors are generalized manifolds. Further, Wilder [**2**] says these factors are manifolds; thus, M is too. Clearly, then a 3-gm which is locally a product space is also a manifold.

Efforts have been made to weaken this hypothesis. In [**3**] K.W. Kwun and F. Raymond proved that a 3-gm which is locally conical is a manifold. M is locally conical if for all P in M, P has a neighborhood N such that $N-P=E^1\times bN$. Our results show that one need not specify the factors in advance. By assuming, only, that $N-P$ is any product space, we can show that M is still a manifold; see Theorem 3.

LEMMA 1. *If M is a connected m-gm, for $m\geqq 2$ such that for P in M, $M-P=A\times B$ is a product space, then each of $M-P$, A, and B is homologically trivial up through dimension $m-2$.*

PROOF. Proof of these claims is exactly analogous to Chapter 3 of the author's dissertation [**4**], except for one minor change. We replace the fact that $\pi_k A\times B=\pi_k A\times\pi_k B$ with the Künneth formula and an induction on k. It is also still true, that if B is compact, then $M-P=E^1\times B$.

Let M be a connected 3-gm.

THEOREM 2. *For P in M, if $M-P$ is a product space, then M is either S^3 or E^3.*

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PROOF. By Lemma 1, the factorization of $M-P$ is either $E^1 \times E^2$ or $E^1 \times S^2$, respectively.

THEOREM 3. *If for each P in M, there is an open neighborhood N of P such that $N-P$ is a product space, then M is a classical 3-manifold.*

PROOF. If N is compact, then $N=M$ and $M=S^3$, by Theorem 2. If N is not compact, then $N=E^3$ by Theorem 2.

Now, let M be a connected m-gm, for $m \geqq 4$.

THEOREM 4. *If for P in M, $M-P$ is a product space, then either $M-P$ is homologically trivial or $M-P$ is E^1 times a generalized $(m-1)$-sphere.*

PROOF. By Lemma 1, each of $M-P=A \times B$, and A, and B is $(m-2)$-connected with respect to homology. If neither A nor B is compact, then each is homologically trivial and of course so is $M-P$. If B is compact, then as mentioned in the proof of Lemma 1, we have $M-P=E^1 \times B$. Since B is closed and $(m-2)$-connected (homology) the theorem follows.

THEOREM 5. *If for P in M, $M-P$ is a product of (many?) factors each of dimension 2 or less, then $M=S^m$.*

PROOF. In view of Lemma 1, we may assume that none of the factors is compact. According to Theorem 4, each factor is either E^1 or E^2.

Next, let M be a compact connected m-gm, for $m \geqq 5$.

THEOREM 6. *For P in M, if $M-P$ is a product of simply-connected (homotopy fundamental group is trivial) PL manifolds A and B, then $M=S^m$.*

PROOF. Using Theorem 4, $M-P$, A, and B are each $(m-2)$-connected (homotopy this time!). If neither A nor B is compact, then each is contractible.

J. Stallings [5] proved that in this case $M-P=E^m$. Finally, let M be a connected m-gm, for $m \geqq 6$.

THEOREM 7. *If for all P in M, there is a neighborhood N of P such that $N-P=A \times B$ is a product of simply-connected manifolds A and B, then M is a classical m-manifold.*

PROOF. By Theorem 4, each of $N-P$, A, and B is $(m-2)$-connected (homotopy!). In light of Theorem 6, we may assume that B is compact. By Lemma 1, $N-P=E^1 \times B$ and B is a homotopy $(m-1)$-sphere. Since $m-1 \geqq 5$ we have $B=S^{m-1}$ by the Poincaré theorem. Thus, $N=E^m$ as desired.

That $N-P=A \times B$ inherits the manifold property from A and B is not new; it is new that the homology groups of A and B may be calculated and need not be assumed. Note that none of the results here or in [4] relies on the unproven Poincaré conjectures.

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