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THE EUCLIDEAN SYMMETRIC ISOSCELES QUEASY FOUR-POINT PROPERTIES

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ABSTRACT. The purpose of this paper is to show that a complete, convex, externally convex metric space is generalized euclidean if and only if it has the euclidean symmetric isosceles queasy four-point property or the euclidean external isosceles queasy four-point property.

1. **Introduction.** Since Wilson [10] characterized generalized euclidean spaces among the class of complete, convex, externally convex metric spaces as those which possess the euclidean four-point property, Blumenthal ([1], [2]) and others have introduced weaker euclidean four-point properties which yield the characterization. Freese [7] introduced an external isosceles feeble euclidean four-point property and proved that it yields a characterization of generalized euclidean space. In [7] Freese asked if an isosceles feeble euclidean four-point property would also provide a characterization. Valentine [9] gave an affirmative answer to his question. Day [5] introduced the euclidean queasy four-point property and showed that it characterizes generalized euclidean space. The purpose of this paper is to consider the symmetric isosceles queasy and external symmetric isosceles queasy four-point properties and show that a complete, convex, externally convex metric space is generalized euclidean if and only if it possesses one of these isosceles four-point properties. It then follows that Valentine's and Freese's characterizations ([9], [7]) are special cases of this work.

2. **The four-point properties.** The following classes of metric quadruples have been introduced by Wilson, Blumenthal and others.

A metric quadruple p_1, p_2, p_3, p_4 belongs to class

C_0 if and only if p_1, p_2, p_3, p_4 are pairwise distinct,

C_1 if and only if p_1, p_2, p_3, p_4 are pairwise distinct and it contains a linear triple,

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C_2 if and only if p_1, p_2, p_3, p_4 are pairwise distinct, p_3 is between p_2, p_4 and $p_2p_3=p_3p_4$,

$C_{3,k}$ if and only if $k>1$ and p_1, p_2, p_3, p_4 are pairwise distinct, p_2 is between p_1 and p_3, while $p_4p_1=p_4p_2$ and $k\cdot p_1p_2=p_1p_3$,

$C_{4,k}$ if and only if $0<k<1$ and p_1, p_2, p_3, p_4 are pairwise distinct, p_3 is between p_1 and p_2, while $p_4p_1=p_4p_2$ and $k\cdot p_1p_2=p_1p_3$.

DEFINITIONS. A metric space has the euclidean, euclidean weak, euclidean feeble, euclidean k-external isosceles feeble, euclidean k-isosceles feeble four-point property provided every quadruple of its points of class C_0, C_1, C_2, $C_{3,k}$, $C_{4,k}$, respectively, is congruently embeddable in euclidean space. A metric space is said to have the euclidean queasy four-point property provided for each two of its points p and r there exists a point q between them such that for all points s of the space the quadruple p, q, r, s is congruently embeddable in euclidean space. A metric space is said to have the euclidean symmetric isosceles queasy four-point property provided for each pair of its distinct points p and r there is a number $\lambda=\lambda(p,r)$ $(0<\lambda<1)$ such that if q is a point between p and r such that $pq=\lambda pr$ or $qr=\lambda pr$, then for each point s in the space for which $ps=rs$ the quadruple p, q, r, s is congruently embeddable in euclidean space.

A metric space is said to have the euclidean external symmetric isosceles queasy four-point property provided for each pair of its distinct points p and r there is a number $\lambda=\lambda(p,r)$ $(1<\lambda<k$, k arbitrary but fixed) such that if q is a point for which r is between p and q and $pq=\lambda pr$ or p is between q and r and $qr=\lambda pr$, then for each point s in the space for which $ps=rs$ the quadruple p, q, r, s is congruently embeddable in euclidean space.

3. The euclidean symmetric isosceles queasy four-point property.

From [9] it is known that the euclidean isosceles feeble four-point property (with $k=\frac{1}{2}$) characterizes generalized euclidean space among the class of complete, convex, externally convex metric spaces. In order to show the euclidean symmetric isosceles queasy four-point property effects such a characterization it suffices to show that a metric space possessing this property also possesses the euclidean isosceles feeble four-point property $(k=\frac{1}{2})$. In the discussion that follows, M will denote a complete, convex, externally convex metric space that possesses the euclidean symmetric isosceles queasy four-point property.

LEMMA 3.1. *Each two distinct points of M are endpoints of exactly one metric segment.*

PROOF. Since M is complete and convex, each two points of M are endpoints of at least one segment. Assume for an indirect proof that M

contains points p_1 and p_3 and distinct points p_4 and p_4^* which are midpoints of p_1 and p_3. By the euclidean symmetric isosceles queasy four-point property there is a point p_2 between p_1 and p_3 such that each of the quadruples p_1, p_2, p_3, p_4 and p_1, p_2, p_3, p_4^* is congruently embeddable in euclidean space. Since p_1, p_2, p_3; p_1, p_3, p_4; and p_1, p_3, p_4^* are linear triples, by the two-triple property for euclidean space, it follows that p_2, p_4^*, p_3; p_2, p_4, p_3; p_1, p_2, p_4^*; and p_1, p_2, p_4 are also linear triples. Since p_4 and p_4^* are midpoints of p_1 and p_3, it is easily seen that $p_2p_4=p_2p_4^*$ and no generality is lost in assuming $p_2p_4p_3$ holds. The euclidean symmetric isosceles queasy four-point property implies the existence of a point p between p_4 and p_4^* such that the quadruples p_2, p_4, p_4^*, p and p_3, p_4, p_4^*, p are congruently embeddable in euclidean space and consequently $pp_2<p_2p_4$ and $pp_3<p_3p_4$ so $pp_2+pp_3<p_2p_4+p_4p_3=p_2p_3$, contrary to the triangle inequality. Thus for each pair of distinct points of M, M contains exactly one midpoint and hence each pair of distinct points lie on a unique metric segment.

LEMMA 3.2. *Metric segments in M admit unique prolongations.*

PROOF. Since M is externally convex, the segment $S(p, q)$ admits a prolongation. Suppose $S(p, q)$ has two prolongations through q. Let s be a point on one prolongation and t a point on the other prolongation such that $qp=qs=qt$. Let r denote a point of $S(s, t)$ for which the quadruples p, s, r, t and q, s, r, t are embeddable in the euclidean plane. Applying the law of cosines to the triples p, s, t and p, s, r to obtain $\cos\sphericalangle s{:}t, p$ we obtain

$$(1)\qquad (ps^2 + st^2 - pt^2)/2\cdot ps\cdot st = (ps^2 + rs^2 - pr^2)/2\cdot ps\cdot rs.$$

In a similar manner we obtain $\cos\sphericalangle s{:}t, q$ in two ways and we have

$$(2)\qquad (qs^2 + st^2 - qt^2)/2qs\cdot st = (qs^2 + rs^2 - qr^2)/2qs\cdot rs.$$

If $\lambda\cdot st$ is substituted for rs in (1) and (2) and the resulting equations are solved for $\lambda\cdot st^2$ it follows that

$$(3)\qquad 3qs^2 = pr^2 - qr^2.$$

However, by the triangle inequality $pr\leqq qs+qr$ and consequently $3qs^2=pr^2-qr^2<qs^2+2\cdot qs\cdot qr<qs^2+2qs^2=3qs^2$, since q, r, s, t are congruently embeddable in the euclidean plane implies $qr<qs$. This contradiction completes the proof.

The results of Lemmas 3.1 and 3.2 may now be combined to obtain

LEMMA 3.3. *Each two distinct points of M are contained in exactly one line of M.*

LEMMA 3.4. *The space M possesses the euclidean isosceles feeble four-point property* $(k=\frac{1}{2})$.

PROOF. Suppose p, q, r are points of M such that $pq=pr$, and suppose m is the midpoint of q and r. By the euclidean symmetric isosceles queasy four-point property M contains a point q_1 between q and r such that $\lambda qr=qq_1$ and p, q, q_1, r are congruently embeddable in the euclidean plane. In the event that $\lambda=\frac{1}{2}$ the proof is complete. If $\lambda\neq\frac{1}{2}$, then as above a point r_1 exists such that $\lambda qr=rr_1$ and p, q, r_1, r are congruently embeddable in the euclidean plane. Since $qq_1=rr_1$, it follows from the euclidean law of cosines that $pq_1=pr_1$. Let $K(q, r)=\{\lambda \in (0, 1) \mid \text{there exist points } x \text{ in } S(q, m), y \text{ in } S(m, r) \text{ such that } qx/qr=ry/qr=\lambda \text{ and } px=py\}$. If $\lambda(q, r)\neq\frac{1}{2}$, then $K(q, r)$ is a nonempty subset of real numbers bounded above by $\frac{1}{2}$ and hence lub $K(q, r)$ exists and is less that or equal to $\frac{1}{2}$. If lub $K(q, r)=\frac{1}{2}$, then for each n, $S(q, r)$ contains points q_n, r_n such that $q_n m r_n$ holds, $qq_n/qr=rr_n/qr>\frac{1}{2}-1/n$ and $pq_n=pr_n$, and so $\lim q_n=m$, $\lim r_n=m$. It follows from the continuity of the metric that p, q, r, m are congruently embeddable in the euclidean plane. If lub $K(q, r)<\frac{1}{2}$, then $S(q, m)$, $S(m, r)$ contain points x and y, respectively, such that the quadruple p, q, r, x; p, q, r, y; p, x, y, m are congruently embeddable in the euclidean plane and $px=py$, $qx=ry$. Applications of the euclidean law of cosines yields p, q, r, m are congruently embeddable in the euclidean plane.

THEOREM 1. *A complete, convex, externally convex metrix space is generalized euclidean if and only if it possesses the euclidean symmetric isosceles queasy four-point property.*

PROOF. By Lemma 3.4 the euclidean symmetric isosceles queasy four-point property is equivalent to the euclidean isosceles feeble four-point property $(k=\frac{1}{2})$. Therefore, by a result of **[9]**, the conclusion follows.

Clearly the euclidean k-isosceles feeble four-point property implies the euclidean symmetric isosceles queasy four-point property and we thus have

THEOREM 2. *A complete, convex, externally convex metric space is generalized euclidean if and only if it possesses the euclidean k-isosceles feeble four-point property.*

It should be noted that it follows as in **[5]** that if a complete, externally convex, metric space possesses the euclidean symmetric isosceles queasy (feeble) four-point property and contains for each pair of its distinct points q and r a point p such that $pq=pr$, then the metric space is convex.

4. **The euclidean external symmetric isosceles queasy four-point property.** As was noted in the introduction, Freese **[7]** characterized generalized

euclidean space using the euclidean k-external isosceles feeble four-point property with $k=\frac{3}{2}$. A careful examination of the proofs of his lemmas shows that only minor changes need to be made to obtain the results using the euclidean external symmetric isosceles queasy four-point property. We thus note the following two theorems.

THEOREM 3. *A complete, convex, externally convex metric space is generalized euclidean if and only if it possesses the euclidean external symmetric isosceles queasy four-point property.*

THEOREM 4. *A complete, convex, externally convex metric space is generalized euclidean if and only if it possesses the euclidean k-external isosceles feeble four-point property.*

As in [5] the euclidean external symmetric isosceles queasy and the euclidean k-external isosceles feeble four-point properties imply the space is externally convex provided for each pair of distinct points q, r there is a point p such that $pq=pr$.

5. **Concluding remark.** The euclidean isosceles queasy four-point property may be stated as follows.

A metric space M possesses the euclidean isosceles queasy four-point property provided for each pair of its distinct points p, r there is a point q between p and r such that for each point s of M whenever $ps=rs$, the quadruple p, q, r, s is congruently embeddable in euclidean space.

It should be noted that the difference between the symmetric queasy and the queasy four-point properties is that with the exception when $\lambda=\frac{1}{2}$ the symmetric queasy insures the existence of at least two points q, q' between distinct points p and r such that for each s of the space, $ps=rs$ implies the quadruples p, q, r, s and p, q', r, s are embeddable in the euclidean plane. In this work we only used the stronger form in the proof of Lemma 3.4. It would be interesting to know if the euclidean isosceles queasy four-point property effects such a characterization. If it does, this would not only be a strengthening of our result, but it would also strengthen the result of Day [5]. Similarly it would be interesting to know if the symmetric property of the euclidean external isosceles queasy four-point property could be deleted. This property is not needed until Freese's Lemma 6.

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