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EMBEDDING NUCLEAR SPACES IN PRODUCTS OF AN ARBITRARY BANACH SPACE

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ABSTRACT. It is proved that if E is an arbitrary nuclear space and F is an arbitrary infinite-dimensional Banach space, then there exists a fundamental (basic) system $\mathscr{V}$ of balanced, convex neighborhoods of zero for E such that, for each V in $\mathscr{V}$, the normed space E_V is isomorphic to a subspace of F. The result for $F=l_p$ $(1 \leqq p \leqq \infty)$ was proved by A. Grothendieck.

This paper is an outgrowth of an interest in varieties of topological vector spaces [**2**] stimulated by J. Diestel and S. Morris, and is in response to their most helpful discussions and questions. The main theorem, valid for arbitrary infinite-dimensional Banach spaces, was first proved by A. Grothendieck [**3**] (also, see [**5**, p. 101]) for the Banach spaces l_p $(1 \leqq p \leqq \infty)$ and later by J. Diestel for the Banach space c_0.

Our demonstration relies on two profound results of T. Kōmura and Y. Kōmura [**4**] and C. Bessaga and A. Pełczyński [**1**], respectively:

(i) A locally convex space is nuclear if and only if it is isomorphic to a subspace of a product space $(s)^I$, where I is an indexing set and (s) is the Fréchet space of all rapidly decreasing sequences.

(ii) Every infinite-dimensional Banach space contains a closed infinite-dimensional subspace which has a Schauder basis.[1]

Recall that for a balanced, convex neighborhood V of zero in a locally convex space E, E_V is a normed space which is norm-isomorphic to $(M, p|_M)$, where p is the gauge of V and M is a maximal linear subspace of E on which p is a norm; $\tilde{E}_V$ is the completion of E_V. Denote by (s) the nuclear Fréchet space of rapidly decreasing sequences, so that

$$(s) = \left\{(\lambda_n) : \sup_n |n^k \lambda_n| < \infty,\ k = 1, 2, \cdots\right\},$$

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[1] This result was stated without proof in Banach's book (1932).

with the system of seminorms

$$\left\{p_k((\lambda_n)) = \sup_n |\, n^k\lambda_n \,| : k = 1, 2, \cdots\right\}.$$

LEMMA 1. *Let F be an arbitrary infinite-dimensional Banach space with a Schauder basis, and let U be a neighborhood of zero in (s). There exists a balanced, convex neighborhood of zero $V \subset U$ such that $\widetilde{(s)}_V$ is norm-isomorphic to F.*

PROOF. Let $\{x_n\}$ be a Schauder basis for F with coefficient functionals $\{f_n\} \subset F'$, and with $\|x_n\| = 1$ $(n = 1, 2, \cdots)$. By the uniform boundedness theorem, the (absolute) polar $\{f_n\}^0$ of $\{f_n\}$ is a neighborhood of zero, and its gauge p is thus a continuous seminorm. Therefore $K\|\sum_{n=1}^\infty a_n x_n\| \geqq p(\sum_{n=1}^\infty a_n x_n) = \sup_n |a_n|$ for each $\sum_{n=1}^\infty a_n x_n \in F$, where K is some positive constant. For some $\varepsilon > 0$ and positive integer k,

$$V' = \{x \in (s) : p_k(x) \leqq \varepsilon\} \subset U.$$

Define a norm q on (s) so that, for each $(\lambda_n) \in (s)$,

$$\begin{aligned} K\left(\sum_{n=1}^\infty n^{-2}\right) p_{k+2}((\lambda_n)) &= K\left(\sum_{n=1}^\infty n^{-2}\right) \sup_n |n^{k+2}\lambda_n| \geqq K \sum_{n=1}^\infty |n^k \lambda_n| \\ (*) \qquad &= K \sum_{n=1}^\infty \|n^k \lambda_n x_n\| \geqq K \left\| \sum_{n=1}^\infty n^k \lambda_n x_n \right\| \\ &\equiv q((\lambda_n)) \geqq p\left(\sum_{n=1}^\infty n^k \lambda_n x_n\right) = p_k((\lambda_n)). \end{aligned}$$

Therefore $V = \{x \in (s) : q(x) \leqq \varepsilon\} \subset V' \subset U$ is a neighborhood of zero, and $(s)_V$ is norm-isomorphic to $((s), q)$, which is clearly norm-isomorphic to a dense subspace of F by $(*)$.

REMARK. It is clear from the proof that the map

$$(\lambda_n) \rightarrow \left(\sum_{n=1}^\infty n^k \lambda_n x_n\right)_{k=1}^\infty$$

embeds (s) in the product space $F \times F \times F \times \cdots$, and thus by (i) and (ii), each nuclear space can be embedded in some product of any given infinite-dimensional Banach space.

LEMMA 2. *The space (s) is isomorphic to $(s) \times (s)$.*

PROOF. The map $(\lambda_n) \rightarrow ((\lambda_{2n-1}), (\lambda_{2n}))$ is a linear, bicontinuous bijection from (s) onto $(s) \times (s)$.

THEOREM. *Let E be an arbitrary nuclear space and F an arbitrary infinite-dimensional Banach space. Each neighborhood U of zero in E contains a balanced, convex neighborhood V of zero such that $\tilde{E}_V$ is norm-isomorphic to a subspace of F.*

PROOF. Let I be any indexing set, and U a neighborhood of zero in the product space $E=(s)^I$. Thus there exists a finite subset A of I such that $U\supset W\times(s)^B$, where $B=I\sim A$ and W is a neighborhood of zero in $(s)^A$. By (ii) there exists a closed infinite-dimensional subspace F_0 of F which has a Schauder basis, and by Lemmas 1 and 2, there is a balanced, convex neighborhood of zero $V\subset W$ such that the completion of $((s)^A)_V$ is norm-isomorphic to F_0. Now $V'=V\times(s)^B$ is a neighborhood of zero contained in U such that $E_{V'}$ is norm-isomorphic to $((s)^A)_V$ and the conclusion of the theorem holds for $E=(s)^I$. But then the conclusion clearly holds for any subspace of $(s)^I$, and hence for any nuclear space E, by (i).

COROLLARY. *The variety* ([2], [2a]) *generated by any infinite-dimensional Banach space contains the variety of all nuclear spaces.*

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