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ON THE ACCUMULATION OF THE ZEROS OF A BLASCHKE PRODUCT AT A BOUNDARY POINT

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ABSTRACT. Let B be a Blaschke product with zeros $\{a_n\}$. The series $\sum (1-|a_n|^2)/|1-\bar{\zeta}a_n|^\gamma$, where $\gamma \geqq 1$ and $|\zeta|=1$, has been used by P. R. Ahern, D. N. Clark, G. T. Cargo, and others in the study of the boundary behavior of B and various associated functions. In this paper the convergence of this series is shown to be equivalent to a condition on a reproducing kernel for a subspace of the Hardy space H^2. Some related conditions and corollaries are also given.

1. **Introduction.** Let U denote the unit disc in the complex plane. For $0<|a_n|<1$ and $\sum (1-|a_n|)<\infty$, put

$$B(z) = \prod_{n=1}^{\infty} \frac{\bar{a}_n}{|a_n|} \frac{a_n - z}{1 - \bar{a}_n z}.$$

B is holomorphic in U and is called a Blaschke product. If $\gamma \geqq 1$ and if ζ is any point on the boundary, ∂U, of U, we can associate with the sequence $\{a_n\}$, the series

$$\sum_{n=1}^{\infty} (1 - |a_n|^2)/|1 - \bar{\zeta}a_n|^\gamma. \tag{1}$$

In [1] and [3], Ahern and Clark study properties necessary and sufficient for the convergence of (1) for integer values of γ. For general γ, Cargo [4] has given a necessary condition for the convergence of (1) in terms of the existence of certain tangential limits for B and all its subproducts. This condition is shown by Linden and Somadasa in [7] to be not sufficient. A related condition, which is also necessary but not sufficient, can be found in [8]. In §3 of this paper, we give, for any $\gamma \geqq 1$, a necessary and sufficient condition for the convergence of (1). We also give the corresponding result for general inner functions.

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In §4, a corollary is given which is motivated by some work on the segmental variation of Blaschke products by Cargo [5]. Also, we present another necessary and sufficient condition for the convergence of (1) when γ is an integer. §5 is devoted to an application, observed by Professor D. N. Clark, of the results of §3.

2. **Preliminaries.** For each $\zeta \in \partial U$, each $\gamma \geqq 1$, and each $m>0$, let $\Gamma=\Gamma_{\zeta,\gamma,m}$ be the "curve" in U given by $\Gamma(\theta)=(1-m|\theta|^\gamma)e^{i\theta}$ for $0<|\theta|<\min\{\pi, m^{-1/\gamma}\}$. Γ encloses the region $R(m, \zeta, \gamma)$ which was introduced by Cargo in his study of tangential limits [4].

Let H^2 denote the usual Hardy class of functions holomorphic in U. A well-known theorem of Beurling states that any closed invariant subspace of H^2 is of the form ϕH^2, where ϕ is an inner function. That is, ϕ is a bounded holomorphic function in U whose radial limits are of modulus one at almost all points of ∂U. If ϕ is an inner function, then $\phi=MBs$ where M is a monomial, B is a Blaschke product with zeros $\{a_n\}$ $(a_n\neq 0)$, and s is a singular inner function. By this we mean

$$s(z) = \exp\left\{-\int_0^{2\pi} (e^{it}+z)/(e^{it}-z)\, d\sigma(t)\right\},$$

where σ is a finite nonnegative singular measure. The function s, in turn, can be factored into the product of two singular inner functions, s_c and s_a, corresponding to, respectively, a continuous measure σ_c and a purely atomic measure σ_a with $\sigma=\sigma_c+\sigma_a$. For a general discussion of the above see, for example, [9].

For any inner function ϕ, we let $(\phi H^2)^\perp$ be the orthogonal complement of ϕH^2 in H^2. For each $z \in U$, define the function $K_z=K_z^\phi$ in U by

$$K_z(\lambda) = (1-\overline{\phi(z)}\phi(\lambda))/(1-\bar{z}\lambda).$$

It is observed in [1] and [2] that $K_z \in (\phi H^2)^\perp$ and that $(f, K_z)=f(z)$ for each $f \in (\phi H^2)^\perp$. In particular, we have

$$\|K_z\|^2 = K_z(z) = (1-|\phi(z)|^2)/(1-|z|^2),$$

where $\|K_z\|$ stands for the H^2 norm of K_z.

3. **The main result.** We begin with the case of a Blaschke product.

THEOREM 1. *Let B be a Blaschke product with zeros $\{a_n\}$, and choose any $\zeta \in \partial U$, $\gamma \geqq 1$, and $m>0$. Then* (1) *converges if and only if*

$$\int_{\Gamma_{\zeta,\gamma,m}} \|K_z\|^2\, |dz| < \infty. \tag{2}$$

PROOF. In [1], it is shown that

$$\|K_z\|^2 = \sum_n |B_n(z)|^2 (1 - |a_n|^2)/|1 - \bar{a}_n z|^2,$$

where B_n is the subproduct of B with zeros $a_1, \cdots, a_{n-1}$. The theorem is then obvious for any finite Blaschke product, and so we shall concern ourselves only with the case in which B is an infinite Blaschke product. Also, we see by replacing B with the Blaschke product with zeros $\{\bar{\zeta} a_n\}$ that we can assume that $\zeta = 1$ without any loss in generality.

Now, let us assume that (1) converges. This implies, since $\zeta = 1$, that a_n can be real and positive for at most finitely many n. So, we can assume that arg a_n is never zero, where by arg a_n we mean the value of the argument of a_n which is in the interval $(-\pi, \pi]$. Then, it is shown in [4] that the convergence of (1) implies that

$$\sum_{n=1}^{\infty} (1 - |a_n|)/|\arg a_n|^\gamma \tag{3}$$

converges.

Let $c = \min\{\pi, m^{-1/\gamma}\}$. We have, for $\Gamma = \Gamma_{1,\gamma,m}$,

$$\int_0^c \|K_{\Gamma(\theta)}\|^2 \, d\theta \leqq \sum_n 2(1 - |a_n|) \int_0^c |1 - \bar{a}_n \Gamma(\theta)|^{-2} \, d\theta.$$

Set $t_n = \arg a_n$. By the law of cosines,

$$\begin{aligned} |1 - \bar{a}_n \Gamma(\theta)|^2 &= (1 - |a_n \Gamma(\theta)|)^2 + 4\,|a_n \Gamma(\theta)| \sin^2 \tfrac{1}{2}(\theta - t_n) \\ &\geqq (1 - |\Gamma(\theta)|)^2 + \tfrac{1}{2}(\theta - t_n)^2 \\ &= m^2 |\theta|^{2\gamma} + \tfrac{1}{2}(\theta - t_n)^2 \end{aligned}$$

for θ close to 0 and any n such that t_n is close to 0 and $|a_n| > \frac{1}{2}$. So, there is a constant c' such that if t_n is small and positive,

$$\begin{aligned} &\int_0^{c'} \frac{d\theta}{|1 - \bar{a}_n \Gamma(\theta)|^2} \\ &\quad \leqq \int_0^{t_n - t_n^\gamma/2} \frac{2\, d\theta}{(\theta - t_n)^2} + \int_{t_n - t_n^\gamma/2}^{t_n + t_n^\gamma/2} \frac{d\theta}{m^2 \theta^{2\gamma}} + \int_{t_n + t_n^\gamma/2}^{c'} \frac{2\, d\theta}{(\theta - t_n)^2} \\ &\quad = 4 t_n^{-\gamma} - 2 t_n^{-1} + [(t_n - t_n^\gamma/2)^{1-2\gamma} - (t_n + t_n^\gamma/2)^{1-2\gamma}]/(2\gamma - 1) m^2 \\ &\qquad + 4 t_n^{-\gamma} - 2/(c' - t_n) \\ &\quad = O(1/t_n^\gamma) \qquad (t_n \to 0), \end{aligned}$$

where the estimate on $[(t_n - t_n^\gamma/2)^{1-2\gamma} - (t_n + t_n^\gamma/2)^{1-2\gamma}]$ can be made by using its binomial expansion. If t_n is close to 0 and negative, we note that $|1 - a_n \Gamma(\theta)| < |1 - \bar{a}_n \Gamma(\theta)|$ for $\theta > 0$ and apply the above to $|1 - a_n \Gamma(\theta)|$.

It then follows from the convergence of (3) that $\int_0^c \|K_{\Gamma(\theta)}\|^2\, d\theta<\infty$. The finiteness of the integral over the interval $(-c, 0)$ is derived analogously. Then (2) follows since $|\Gamma'(\theta)|$ is bounded for $-c<\theta<c$.

Now for the converse, assume that (2) holds. It is easily seen that

$$B'(z) = \sum_{n=1}^{\infty} \tilde{B}_n(z)(1-|a_n|^2)/(1-\bar{a}_n z)^2,$$

where $\tilde{B}_n(z)=B(z)(1-\bar{a}_n z)/(z-a_n)$. Then, $\|K_z\|^2 \geqq |B(z)|\cdot|B'(z)|=\frac{1}{2}|D'(z)|$, where $D=B^2$. So, (2) implies that $\int_\Gamma |D'(z)|\,|dz|<\infty$. Thus, the Blaschke product D has a limit as $z\to 1$ along Γ. Also, (2) implies that

$$\int_\Gamma (1-|D(z)|)/(1-|z|)\,|dz| < \infty.$$

Then if $c=\min\{\pi, m^{-1/\gamma}\}$, $\int_{-c}^{c} (1-|D(\Gamma(\theta))|)\,|\theta|^{-\gamma}\, d\theta<\infty$ since $|\Gamma'(\theta)|$ is bounded away from 0. So, $\sup\{|D(z)|: z\in\Gamma\}=1$, and $|D(z)|\to 1$ as $z\to 1$ along Γ. Thus, $|B(z)|\to 1$ as $z\to 1$ along Γ.

It now follows from (2) and the series expansion for $\|K_z\|$ that

$$\int_{-c}^{c} \sum_n (1-|a_n|^2)/|1-\bar{a}_n\Gamma(\theta)|^2\, d\theta < \infty,$$

if we again use the fact that $|\Gamma'(\theta)|$ is bounded away from 0. This implies that

$$\sum_n (1-|a_n|^2)\int_I |1-\bar{a}_n\Gamma(\theta)|^{-2}\, d\theta < \infty,$$

where $t_n=\arg a_n$, $m'=\min\{m, \frac{1}{2}\}$, and $I=(t_n-m'\,|t_n|^\gamma, t_n)$ if $t_n>0$ and $I=(t_n, t_n+m'\,|t_n|^\gamma)$ if $t_n<0$.

Since $B(z)$ has a limit of modulus one as $z\to 1$ along Γ, a theorem of Lindelöf (see [**6**, p. 460]) says that $B(z)$ has a limit of modulus one as $z\to 1$ inside Γ. So, there is an integer n_0 such that a_n is outside of Γ for each $n\geqq n_0$. In other words, $1-|a_n|<m\,|\arg a_n|^\gamma$ if $n\geqq n_0$. Then,

$$\begin{aligned} |1-\bar{a}_n\Gamma(\theta)|^2 &= (1-|a_n\Gamma(\theta)|)^2 + 4\,|a_n\Gamma(\theta)|\sin^2\tfrac{1}{2}(t_n-\theta) \\ &\leqq (1-|a_n| + m\,|\theta|^\gamma)^2 + (t_n-\theta)^2 \\ &\leqq (m\,|t_n|^\gamma + m\,|\theta|^\gamma)^2 + (t_n-\theta)^2 \leqq 5m^2\,|t_n|^{2\gamma} \end{aligned}$$

if $\theta\in I$ and $n\geqq n_0$. So,

$$\sum_{n=n_0}^{\infty} (1-|a_n|^2)/|\arg a_n|^\gamma < \infty.$$

The convergence of (1) follows since $\sin|\arg a_n|\leqq|1-a_n|$.

Next let us extend Theorem 1 to the case of a general inner function. Since there is much duplication with the proof of Theorem 1, we shall only sketch the proof of the extension.

THEOREM 2. *Let $\phi = MBs$ be an inner function, and choose any $\zeta \in \partial U$, $\gamma \geqq 1$, and $m > 0$. Then* (1) *converges and*

$$\int_0^{2\pi} |1 - \bar{\zeta}e^{it}|^{-\gamma}\, d\sigma(t) \tag{4}$$

converges if and only if

$$\int_{\Gamma_{\zeta,\gamma,m}} \|K_z^{\phi}\|^2\, |dz| < \infty. \tag{5}$$

PROOF. As before, we can assume that $\zeta = 1$. Also, we shall assume, in order to avoid trivialities, that B is an infinite Blaschke product and s has infinitely many point masses and a nontrivial continuous part. Let us note that if $M(z) = bz^k$, $\|K_z^M\|^2 = (1-|z|^{2k})/(1-|z|^2)$ since $|b| = 1$. So, (5) with ϕ replaced by M always holds.

Now let us assume that (1) and (4) converge. Then by Theorem 1, (5) with ϕ replaced by B holds. Since (4) converges, (4) with σ replaced by σ_c converges. In **[2]**, it is shown that

$$\|K_z^{\phi_c}\|^2 = 2\int_0^{\pi} |s_{c,t}(z)|^2/|1 - e^{-it}z|^2\, d\sigma_c(t),$$

where $s_{c,t}$ is the inner function given by

$$s_{c,t}(z) = \exp\left\{-\int_0^t (e^{i\theta} + z)/(e^{i\theta} - z)\, d\sigma_c(\theta)\right\}.$$

The proof that (5) with ϕ replaced by s_c holds now can be seen to parallel the corresponding part of the proof of Theorem 1. Also in **[2]**, there is a series representation for $\|K_z^{\phi_a}\|$, which we can use to prove that (5) with ϕ replaced by s_a holds.

If ϕ and ψ are any two inner functions, then

$$1 - |\phi(z)\psi(z)|^2 = 1 - |\phi(z)|^2 + |\phi(z)|^2\,(1 - |\psi(z)|^2),$$

and so,

$$\|K_z^{\phi\psi}\|^2 = \|K_z^{\phi}\|^2 + |\phi(z)|^2\, \|K_z^{\psi}\|^2 \leqq \|K_z^{\phi}\|^2 + \|K_z^{\psi}\|^2.$$

Therefore, the results of the last paragraph can be combined to show that (5) holds.

Conversely, assume that (5) holds. If ϕ and ψ are any two inner functions, $1-|\phi(z)\psi(z)|^2 \geqq 1-|\phi(z)|^2$, and so $\|K_z^{\phi\psi}\|^2 \geqq \|K_z^{\phi}\|^2$. Thus, (5) holds with ϕ replaced by B, s_c, or s_a. Therefore, (1) converges, and in an analogous way (4) with σ replaced by σ_c or σ_a converges.

4. **The variation of an inner function.** In [5], Cargo proves that the convergence of (1) with $\gamma=1$ implies the finiteness of the length of the curve which is the image under B of the line segment joining any point $a \in U$ with ζ. This leads us to

COROLLARY. *Let $\phi=MBs$ be an inner function, and choose any $\zeta \in \partial U$, $\gamma \geqq 1$, and $m>0$. If* (1) *and* (4) *converge, then*

$$\int_{\Gamma_{\zeta,\gamma,m}} |\phi'(z)|\,|dz| < \infty. \tag{6}$$

PROOF. As we noted in the proof of Theorem 1, $\|K_z^B\|^2 \geqq |B(z)|\cdot|B'(z)|$. It is shown in [4] that the convergence of (1) implies that $|B(z)|\to 1$ as $z\to\zeta$ along Γ. So, (6) with ϕ replaced by B follows from (2). In a similar way, we can show that (6) also holds with ϕ replaced by s_c or s_a. The desired result then follows directly.

In [3], Ahern and Clark prove that (1) converges for an integer value of γ if and only if the $(\gamma-1)$st derivative of each subproduct of B, including B itself, has a finite radial limit at ζ. This proves half of

THEOREM 3. *Let B be a Blaschke product with zeros $\{a_n\}$, and choose any $\zeta \in \partial U$ and any positive integer γ. Then* (1) *converges if and only if for each subproduct f of B, including B itself, we have*

$$\int_0^1 |f^{(\gamma)}(\zeta r)|\,dr < \infty. \tag{7}$$

PROOF. If (7) holds for each subproduct f of B, we can use the just mentioned result of Ahern and Clark to immediately give the convergence of (1). So, let us now assume that (1) converges. As usual we can take ζ to be 1. So we can assume that (3) converges.

In [1], it is noted that

$$B^{(\gamma)}(r) = \sum_{j=0}^{\gamma-1} \binom{\gamma-1}{j} \sum_{n=1}^{\infty} \tilde{B}_n^{(\gamma-1-j)}(r)\, \frac{(j+1)!\,\bar{a}_n^j(1-|a_n|^2)}{(1-\bar{a}_n r)^{j+2}}.$$

Also from [1], we see that we can find a constant C_0 such that $|\tilde{B}_n^{(\gamma-1-j)}(r)| \leqq C_0$ for any $0<r<1$, $j=0, 1, \cdots, \gamma-1$, and $n=1, 2, \cdots$. Then

$$\int_0^1 |B^{(\gamma)}(r)|\,dr \leqq C_1 \sum_{n=1}^{\infty} (1-|a_n|^2) \int_0^1 |1-\bar{a}_n r|^{-(\gamma+1)}\,dr$$

for some constant C_1. By the law of cosines, $|1-\bar{a}_n r|^2 \geqq (1-r)^2 + \frac{1}{4}|t_n|^2$ for $t_n = \arg a_n$ close to 0. Then for any such n, $|1-\bar{a}_n r|^{\gamma+1} \geqq (1-r)^{\gamma+1} + k^{-1}|t_n|^{\gamma+1}$, $k=2^{\gamma+1}$. So,

$$\int_0^1 |1-\bar{a}_n r|^{-(\gamma+1)}\,dr \leqq \int_0^{1-|t_n|} (1-r)^{-(\gamma+1)}\,dr + \int_{1-|t_n|}^1 k\,|t_n|^{-(\gamma+1)}\,dr$$
$$= (k+1/\gamma)\,|t_n|^{-\gamma} - 1/\gamma.$$

Therefore, the convergence of (3) implies that (7) holds for $f=B$. If f is any other subproduct of B, the zeros of f will satisfy $\sum (1-|a_n|^2)/|1-a_n|^\gamma < \infty$, and so we can apply the above work directly to f to show that (7) holds for f.

It is possible to extend Theorem 3 to general inner functions by introducing the concept of divisor as was done in [**3**].

5. **An application.** Let ϕ be any inner function, and let μ be any complex number of modulus less than 1. It is easily checked that $\phi_\mu(z) = (\phi(z)-\mu)/(1-\bar{\mu}\phi(z))$ is again an inner function.

THEOREM 4. *Let $\phi = MBs$ be an inner function, and let μ be any complex number of modulus less than* 1. *If, for any $\zeta \in \partial U$ and any $\gamma \geqq 1$,* (1) *and* (4) *converge, then the corresponding sum and integral for ϕ_μ also converge.*

PROOF. Let us compute

$$\begin{aligned}\|K_z^{\phi_\mu}\|^2 &= (1-|\phi(z)-\mu|^2\,|1-\bar{\mu}\phi(z)|^{-2})/(1-|z|^2)\\ &= |1-\bar{\mu}\phi(z)|^{-2}(|1-\bar{\mu}\phi(z)|^2 - |\phi(z)-\mu|^2)/(1-|z|^2)\\ &= |1-\bar{\mu}\phi(z)|^{-2}((1-|\mu|^2)-(1-|\mu|^2)\,|\phi(z)|^2)/(1-|z|^2)\\ &= (1-|\mu|^2)\,|1-\bar{\mu}\phi(z)|^{-2}\,\|K_z^\phi\|^2.\end{aligned}$$

Since (1) and (4) converge for ϕ, Theorem 2 implies that (5) holds for ϕ. But since $|\mu|<1$, the above computation implies that (5) holds for ϕ_μ. Again applying Theorem 2, we see that (1) and (4) converge for ϕ_μ.

The above theorem gives us information about the μ points of a Blaschke product B. Let $\{b_n\}$ be the sequence of all points (counting multiplicity) for which $B(b_n)=\mu$. Choose any $\zeta \in \partial U$ and any $\gamma \geqq 1$. If (1) converges, then (1) with the sequence $\{a_n\}$ replaced by the sequence $\{b_n\}$ converges by Theorem 4 since $\{b_n\}$ is the sequence of zeros of B_μ.

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