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THE SPACE OF RETRACTIONS OF A 2-MANIFOLD

NEAL R. WAGNER

ABSTRACT. Let M^2 be a 2-manifold and let Λ be the embedding of M^2 into its space of retractions which maps each point to the constant retraction to that point. Denote by $\mathscr{L}(M^2)$ the component containing the image of Λ. The embedding Λ, with range restricted to $\mathscr{L}(M^2)$, is shown to be a weak homotopy equivalence if M^2 is compact, or if M^2 is complete and the metric topology is used.

1. **Introduction.** In **[8]**, the author studied one component of the space of retractions of the 2-sphere and the annulus, namely the component consisting of retractions with contractible image (the nullhomotopic retractions). This paper extends the results of **[8]** to more general 2-manifolds. The techniques are similar to those in **[8]**, and we refer the reader to that paper. The author would like to express his gratitude to the referee of **[8]** for suggesting the method of proof in this paper and to C. W. Neville for some helpful conversations.

For any 2-manifold M^2, let $\mathscr{R}(M^2)$ denote the space of retractions of M^2, with either the compact-open or the sup-metric topology. Let Λ be the embedding of M^2 into $\mathscr{R}(M^2)$ which takes each $u \in M^2$ to the constant retraction of M^2 to u. Denote by $\mathscr{L}(M^2)$ the component of $\mathscr{R}(M^2)$ containing the image of Λ. We can now state the main result of this paper.

THEOREM. *The embedding* $\Lambda: M^2 \to \mathscr{L}(M^2)$ *is a weak homotopy equivalence in two cases*:

(1) *if* M^2 *is compact and the compact-open* (=*sup-metric*) *topology is used*, *or*

(2) *if* M^2 *is complete and the sup-metric topology is used.*

These cases are exactly those for which $\mathscr{L}(M^2)$ consists of all retractions with compact, contractible image (see §3). Any (second countable)

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2-manifold is complete in some metric, so case (2) of the theorem will apply in general with respect to some sup-metric on $\mathscr{R}(M^2)$.

By definition [7, p. 404], Λ is a weak homotopy equivalence if the induced maps $\Lambda_* : \pi_n(M^2) \to \pi_n(\mathscr{L}(M^2))$ are isomorphisms, for each n. If ev denotes the evaluation map ($\mathrm{ev}(\varphi) = \varphi(u_0)$ for any retraction φ and some basepoint u_0 of M^2), then clearly $\mathrm{ev} \circ \Lambda$ is the identity map on M^2, so that it will suffice to prove that Λ_* is surjective. The underlying idea of the proof is to produce, corresponding to certain retractions φ, a canonical simple closed curve in M^2 which bounds a disk containing the image of φ. Then, essentially working within the disk, we can construct a homotopy from φ to $\Lambda \circ \mathrm{ev}(\varphi)$. These simple closed curves will be provided by a selection theorem of E. Michael. A finite-dimensionality condition in this theorem prevents us from simply showing that $\Lambda \circ \mathrm{ev}$ is homotopic to the identity map on $\mathscr{L}(M^2)$ and concluding that Λ is a homotopy equivalence.

2. **Preliminaries on conformal mapping.** In any 2-manifold M^2, let $\mathscr{S}$ denote the collection of simple closed curves which bound a disk in M^2 and which do not meet ∂M^2. (Since the theorem was proved in [**8**] for S^2, we assume M^2 is not S^2.) We shall use much of the notation of [**8**], which included E^2 for Euclidean 2-space, B^2 for the unit 2-ball, and C_r for the circle with center at the origin and radius $r > 0$. In addition, we denote the closed disk which $J \in \mathscr{S}$ bounds by $B(J)$, and its interior by $\mathrm{int}(B(J))$. If $K \in \mathscr{S}$ and $J \subset \mathrm{int}(B(K))$, then let $A(J, K)$ denote the closed annular region bounded by J and K. In particular, we use A^2 for $A(C_1, C_2)$.

The topology of 0-regular convergence can be defined on $\mathscr{S}$ as follows: a sequence $\{J_i\}$ converges *0-regularly* to J_0 if there are embeddings $f_i : C_1 \to M^2$ such that $f_i(C_1) = J_i$ and $\{f_i\}$ converges uniformly to f_0 on C_1. (It can be seen from the work below that on $\mathscr{S}$ this definition is equivalent to the usual one. See [**3**] and [**4**].)

LEMMA 1. *Let F be any compact, contractible subset of M^2, not meeting ∂M^2. Then there is a $J \in \mathscr{S}$ such that F is contained in* $\mathrm{int}(B(J))$.

PROOF. The universal covering space of $M^2 \backslash \partial M^2$ will be the plane E^2 or the 2-sphere S^2 [**1**, p. 104]. In either case, F lifts to a homeomorphic copy $\hat{F}$ [7, p. 66], and $E^2 \backslash \hat{F}$ is homeomorphic to E^2 minus a point (and likewise for S^2), so it is easy to produce a family of simple closed curves about $\hat{F}$. One of these which is close enough to $\hat{F}$ will project to a suitable curve about F in M^2, completing the proof.

Let M^2 be any orientable 2-manifold. Give M^2 a conformal structure so as to make it a Riemann surface. (See [**1**] and [**6**].) Suppose that for each nonnegative integer i, elements J_i and K_i of $\mathscr{S}$ are given such that each J_i lies in $\mathrm{int}(B(K_i))$. Then there are homeomorphisms $f_i : B^2 \to B(J_i)$ and

$g_i: A^2 \to A(J_i, K_i)$ which are uniquely determined by the conditions:

(a_1) each f_i is conformal on $\mathrm{int}(B^2)$,

(a_2) each f_i maps the point $(1, 0)$ to some $v_i \in J_i$,

(a_3) each f_i maps the origin to some $u_i \in \mathrm{int}(B(J_i))$,

(b_1) each g_i is a radial contraction of a map conformal on $\mathrm{int}(A(C_1, C_{r(i)}))$, for some $r(i) > 1$, and

(b_2) each g_i maps $(1, 0)$ to v_i.

Moreover, we have the following result.

CONTINUITY PROPERTY. *If $\{J_i\}$ and $\{K_i\}$ converge 0-regularly to J_0 and K_0, respectively, and if $\{u_i\}$ and $\{v_i\}$ converge to u_0 and v_0, respectively, then $\{f_i\}$ converges uniformly to f_0 on B^2 and $\{g_i\}$ converges uniformly to g_0 on A^2.*

PROOF. Select $K \in \mathscr{S}$ such that K_i is contained in $\mathrm{int}(B(K))$ for all sufficiently large i. The set $\mathrm{int}(B(K))$ is an open simply connected Riemann surface, clearly of hyperbolic type, and hence ([**1**, p. 203], [**6**, p. 197]) is conformally equivalent to $\mathrm{int}(B^2)$. This translates the situation to the plane, where the results are known. (See [**3**], [**4**], and [**8**].)

If M^2 is not orientable, it has a conformal structure provided conjugates of conformal maps are included. The above results will still hold if all the maps involved are conformal with respect to an orientation of the domain $\mathrm{int}(B(K))$.

3. **Proof of the theorem.** The proof uses a selection theorem of E. Michael [**5**], which we state here in a very weak form.

THEOREM M. *Let B and X be metric spaces, where B is complete and the dimension of X is $\leqq n$. Let τ be an open mapping of B onto X such that each (point) inverse under τ has vanishing homotopy groups of order $\leqq n-1$, and such that the collection of inverses under τ is equi-LC^{n-1}. Let e be a mapping of a closed subspace Y of X into B such that $e(y) \in \tau^{-1}(y)$, for all $y \in Y$. Then e extends to a mapping e^* of X into B such that $e^*(x) \in \tau^{-1}(x)$, for all $x \in X$.*

(The property equi-LC^{n-1} is a strong form of LC^{n-1}. See [**5**] and [**8**, Definition 1.4].)

Let M^2 be any 2-manifold, and denote by $\mathscr{K}(M^2)$ the space of those retractions of M^2 with compact, contractible image. The construction in this section is designed to handle $\mathscr{K}(M^2)$, and it will follow from later work that $\mathscr{K}(M^2)$ is pathwise connected. If M^2 is compact, there is a number ε such that any two self maps within a distance ε are homotopic. Since the elements of $\mathscr{K}(M^2)$ are nullhomotopic, it is clear that $\mathscr{L}(M^2) = \mathscr{K}(M^2)$. If M^2 is complete and we use the sup-metric topology, then for any $\varphi \in \mathscr{K}(M^2)$, we can produce a compact submanifold N^2 of M^2 and an

$\varepsilon>0$ such that $d(\psi, \varphi)<\varepsilon$ implies $\operatorname{im}(\psi)\subset N^2$, and so for some $\varepsilon'>0$, $d(\psi, \varphi)<\varepsilon'$ implies $\psi \in \mathscr{K}(M^2)$, i.e., $\mathscr{K}(M^2)$ is open in $\mathscr{R}(M^2)$. (We can construct N^2 by covering $\operatorname{im}(\varphi)$ with finitely many suitable closed disks.) Any $\varphi \in \mathscr{R}(M^2)$ at a distance 0 from $\mathscr{K}(M^2)$ must have compact image, since otherwise there would be a sequence of points in $\operatorname{im}(\varphi)$ no two of which are closer together than some $\varepsilon>0$, and then some $\psi \in \mathscr{K}(M^2)$ would not have compact image. Arguing as before, we see that $\varphi \in \mathscr{K}(M^2)$. Thus $\mathscr{K}(M^2)$ is also closed in $\mathscr{R}(M^2)$, so that $\mathscr{L}(M^2)=\mathscr{K}(M^2)$. If M^2 is not compact and the compact-open topology is used, or if M^2 is not complete and the sup-metric topology is used, then $\mathscr{L}(M^2)$ will not be equal to $\mathscr{K}(M^2)$. This can be seen by embedding a half-open interval in M^2 as a closed subspace and considering retractions onto it and onto closed subintervals of it.

As noted in §1, in order to prove the theorem, it suffices to prove that the induced map Λ_* is surjective. Thus let $\Phi:(I^n, \partial I^n)\to(\mathscr{L}(M^2), \Lambda(u_0))$ be an element of $\pi_n(\mathscr{L}(M^2))$, where u_0 is the basepoint of M^2. Recall the definition of $\mathscr{S}$ from §2. For each $\varphi \in \mathscr{L}(M^2)$ such that $\operatorname{im}(\varphi)$ is compact and does not not meet ∂M^2, let $\mathscr{S}(\varphi)$ denote the set of all $J \in \mathscr{S}$ such that $\operatorname{im}(\varphi)$ is contained in $\operatorname{int}(B(J))$. By Lemma 1, each $\mathscr{S}(\varphi)$ is nonempty. In both cases of the theorem, use the elbowroom construction of [**8**, Remark 1.2] to deform φ so that $\operatorname{im}(\varphi)$ is disjoint from ∂M^2 for all $\varphi \in \operatorname{im}(\Phi)$. Let $\mathscr{B}$ be the subspace of $I^n\times\mathscr{S}$ consisting of all (x, J) such that $x \in I^n$ and $J \in \mathscr{S}(\Phi(x))$. Let τ be the restriction to $\mathscr{B}$ of the projection of $I^n\times\mathscr{S}$ onto I^n. Choose a fixed $J_0 \in \mathscr{S}$ such that $u_0 \in \operatorname{int}(B(J_0))$, and let $e:\partial I^n\to\mathscr{B}$ be defined by $e(y)=(y, J_0)$.

We now apply Theorem M with X, Y, B, τ, and e of the theorem equal to I^n, ∂I^n, $\mathscr{B}$, τ, and e as defined in the previous paragraph. In §4 we verify that the hypotheses of Theorem M are satisfied. Assuming this has been done, we get an extension $e^*:I^n\to\mathscr{B}$ such that for each $x \in I^n$, $e^*(x) \in \tau^{-1}(x)=\{x\}\times\mathscr{S}(\Phi(x))$. If τ_2 denotes the projection onto the second coordinate, then $\tau_2 \circ e^*(x)$ is a canonical member of $\mathscr{S}$ such that the disk it bounds contains $\operatorname{im}(\Phi(x))$ in its interior.

For each $x \in I^n$, let $f(x):B(\tau_2 \circ e^*(x))\to B^2$ be the homeomorphism which is conformal on the interior and which maps the point $\operatorname{ev} \circ \Phi(x)= \Phi(x)(u_0)$ to the origin $(0, 0)$. The map $f(x)$ is unique up to a rotation and possibly a reflection (if M^2 is not orientable), but the construction that follows is independent of rotations and reflections. We can define $\Phi(x)$ piecewise, on $B(\tau_2 \circ e^*(x))$ and on the closure of its complement. The map $f(x) \circ \Phi(x) \circ f(x)^{-1}$ is a retraction of B^2, and thus [**8**, Theorem 1.1] is homotopic to the constant retraction to $(0, 0)$. The homotopy is given by $h_t \circ f(x) \circ \Phi(x) \circ f(x)^{-1} \circ h_t^{-1} \circ \rho_t$, where ρ_t projects $A(C_{1-t}, C_1)$ radially to C_{1-t}, and h_t is a radial homeomorphism of B^2 onto $\rho_t(B^2)$. Thus on

$B(\tau_2 \circ e^*(x))$, add $f(x)^{-1}$ on the left and $f(x)$ on the right to get a homotopy from $\Phi(x)$ to $\Lambda(f(x)^{-1}(0, 0))$. On $M^2 \backslash \operatorname{int}(B(\tau_2 \circ e^*(x)))$, use the homotopy $f(x)^{-1} \circ h_t \circ f(x) \circ \Phi(x)$. It is easy to see that these two homotopies agree on the curve $\tau_2 \circ e^*(x)$, so together they give a homotopy from Φ to a map Ψ defined by $\Psi(x) = \Lambda(f(x)^{-1}(0, 0)) = \Lambda \circ \mathrm{ev} \circ \Phi(x)$. Hence Φ is homotopic to $\Lambda \circ \mathrm{ev} \circ \Phi$, so that Λ_* is surjective and Λ is a weak homotopy equivalence.

4. **The hypotheses of Michael's theorem.** In §3, we postponed the verification that the use of Theorem M is justified. It is easy to see that each $\mathscr{S}(\varphi)$ is homeomorphic to an open subspace of $\mathscr{S}$, and if we prove that $\mathscr{S}$ is LC^{n-1}, then the equi-LC^{n-1} property will follow easily, as in [**8**, Lemma 2.3]. Using the local metric on $\mathscr{S}$ described below, the proof that $\mathscr{B}$ is an open subspace of $I^n \times \mathscr{S}$ is similar to that in [**8**, Lemma 2.2], and so the map τ is an open surjection. The remaining hypotheses will be clear from the following lemma.

LEMMA 2. (a) *The space $\mathscr{S}$ can be given a metric in which it is complete.*

(b) *For each $\varphi \in \mathscr{L}(M^2)$ with* $\mathrm{im}(\varphi)$ *not meeting ∂M^2, the space $\mathscr{S}(\varphi)$ is contractible (in itself).*

(c) *The space $\mathscr{S}$ is locally contractible.*

PROOF. Let $J_0 \in \mathscr{S}$ and choose a closed neighborhood of J_0 small enough so that all its elements contain a fixed point u_0 in their interior. For each J in the neighborhood, consider the unique homeomorphism which maps B^2 onto $B(J)$, is conformal on the interior, takes the origin to u_0, and has positive derivative at the origin. (If M^2 is not orientable, it will be unique up to a reflection.) Let $\mathscr{H}$ denote the space of embeddings $f: C_1 \to M^2$ such that $f(C_1) \in \mathscr{S}$. (Give $\mathscr{H}$ the topology of uniform convergence.) The remarks above show that a sufficiently small closed neighborhood in $\mathscr{S}$ is homeomorphic to a closed subspace of $\mathscr{H}$. As in [**8**], $\mathscr{H}$ can be remetrized so as to be complete, so $\mathscr{S}$ can be given a complete metric locally. Also, $\mathscr{S}$ is the continuous image of $\mathscr{H}$ under the obvious projection, so $\mathscr{S}$ is Lindelöf and hence paracompact. Thus $\mathscr{S}$ is metrizable and locally complete, so it can be given a metric which makes it complete [**2**, pp. 190, 236].

To prove part (b), suppose $J_0 \in \mathscr{S}(\varphi)$, and let f be any homeomorphism of $B(J_0) \backslash \mathrm{im}(\varphi)$ onto B^2 minus the origin. For $J \in \mathscr{S}(\varphi)$, let α equal half the supremum of numbers $\beta \in (0, 1]$ such that $f^{-1}(C_\beta)$ lies in $B(J)$. Let $K = f^{-1}(C_\alpha)$. For some $r > 1$, there is a homeomorphism $g: A(K, J) \to A(C_1, C_r)$ which is conformal on the interior and uniquely determined up to a rotation. The collection of curves $\{g^{-1}(C_\beta): 1 \leqq \beta \leqq r\}$ defines a deformation in $\mathscr{S}(\varphi)$ from J to K. We do this simultaneously for all $J \in \mathscr{S}(\varphi)$ to get a deformation of $\mathscr{S}(\varphi)$ onto the set $\{f^{-1}(C_\alpha): 0 < \alpha \leqq \frac{1}{2}\}$. It is then easy to deform this set in $\mathscr{S}(\varphi)$ to the single curve $f^{-1}(C_1) = J_0$.

For part (c), first prove that $\mathscr{H}$ is locally contractible, using techniques similar to those in [8, Lemma 3.4]. (One can work within a fixed disk in M^2, extending an embedding to a homeomorphism and using the fact that the space of homeomorphisms of that disk is locally contractible.) Then use the projection of $\mathscr{H}$ onto $\mathscr{S}$ (and the local inverse homeomorphism) to show that $\mathscr{S}$ is locally contractible.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, EL PASO, TEXAS 79968