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A 3-DIMENSIONAL ABRAHAM-SMALE EXAMPLE[1]

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ABSTRACT. The author constructs an open set of non Ω-stable diffeomorphisms on T^3.

To further the search for generic properties of diffeomorphisms, we construct a new counterexample to the genericity of Axiom A and Ω-stable diffeomorphisms in $\mathrm{Diff}^r(T^3)$, $r \geqq 1$. This paper reduces the dimension in the original counterexample of Abraham and Smale [**1**] by applying a technique of Williams [**3**]. For a general reference, see [**2**].

Let $g_0 : T^2 \to T^2$ be a hyperbolic automorphism with fixed point θ. Let g_t for $t \in [-1, +1]$ be a smooth isotopy of g_0 to "derived-from-Anosov" (D-A) maps g_{-1} and g_1 as constructed in ([**2**], [**3**]). $g_{\pm 1}(\theta) = \theta$, but θ is a source for g_{+1} with 2-dimensional $W^u(\theta, g_1)$ and a sink for g_{-1}. The nonwandering set $\Omega(g_1) = \{\theta\} \cup \Sigma_1$, where Σ_1 is a 1-dimensional attractor; and $\Omega(g_{-1}) = \{\theta\} \cup \Sigma_{-1}$, where Σ_{-1} is a 1-dimensional source. $T^2 = W^u(\theta, g_1) \cup \Sigma_1 = W^s(\theta, g_{-1}) \cup \Sigma_{-1}$. Let U, V be 2-disks in T^2 with $\theta \in \operatorname{int} U \subset \bar{U} \subset \operatorname{int} V \subset W^u(\theta, g_1) \cap W^s(\theta, g_{-1})$ and with $g_{\pm 1}^{\pm 1}(U) \supset V$.

Using g_t, construct diffeomorphism $\bar{G}$ of $T^2 \times [-1, 1]$ such that (i) $T^2 \times \{\pm 1\}$ and $\{\theta\} \times [-1, 1]$ are invariant under $\bar{G}$, (ii) $\bar{G} | T^2 \times \{\pm 1\} = g_{\pm 1}$, (iii) $\bar{G}$ maps points in $U \times [-1, 1]$ closer to $T^2 \times \{+1\} \equiv T^2_{+1}$ and sends points outside $V \times [-1, 1]$ closer to $T^2 \times \{-1\} \equiv T^2_{-1}$. The $W^u(x_{+1})$ and $W^s(x_{-1})$ are now 2-dimensional for $x_{+1} \in \Sigma_{+1}$ and $x_{-1} \in \Sigma_{-1}$. Construct $\bar{G}$ so that for some such x_{+1} and x_{-1}, $W^u(x_{+1})$ and $W^s(x_{-1})$ have a point of transversal intersection. (Alternatively, follow $\bar{G}$ by a map like b below which forces a $W^u(x_{+1})$ to intersect some $W^s(x_{-1})$ transversely.) Since $\Sigma_{\pm 1}$ are topologically transitive, we can write $\Sigma_{-1} \ll \Sigma_{+1}$ as in [**1**]. Extend $\bar{G}$ to a diffeomorphism of $T^2 \times S^1$ with $(\theta, \pm 1) \equiv \theta_{\pm 1}$ and $\Sigma_{\pm 1}$ still hyperbolic. By composing $\bar{G}$ with a C^∞ perturbation with support in $B = [V \backslash U] \times (-1-\delta, -1+\delta)$, one easily obtains G' whose 2-dimensional $W^s(\theta_{-1}, G')$ intersects the 1-dimensional unstable manifolds of Σ_{-1} transversally in B.

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Now, the 1-dimensional stable manifolds from Σ_{+1} foliate $T^2_{+1}\backslash\{\theta_{+1}\}$. Let $G=b\circ G'$ where b has support in $V\times(1-\delta, 1+\delta)$ for small δ and forces the 1-dimensional $W^u(\theta_{-1}, G')$ to intersect the 2-dim set $\bigcup\{W^s(x): x\in\Sigma_{+1}\}$ transversally. Since $W^u(\Sigma_{+1}, G)\pitchfork W^s(\Sigma_{-1}, G)\neq\varnothing$, $W^u(\Sigma_{-1}, G)\pitchfork W^s(\theta_{-1}, G)\neq\varnothing$, and $W^u(\theta_{-1}, G)\pitchfork W^s(\Sigma_{+1}, G)\neq\varnothing$, we have a cycle in Ω. Since Σ_{+1} has type $(1, 2)$ and θ_{-1} has type $(2, 1)$, G does not satisfy Smale's Axiom A [**1**]. Since the W^u and W^s vary continuously and the above intersections are transversal (at least, topologically), all maps sufficiently near G will not satisfy Axiom A. The fact that none of these maps are Ω-stable follows as in §3 of [**1**]. For, the set of H' such that the "first" intersection of $W^s(\theta_{+1}, H')$ with $W^u(\Sigma'_{-1}, H')$ occurs on $W^u(x'; H')$ for some periodic x' in Σ'_{-1} and the set of H'' where the first such intersection occurs at a nonperiodic $W^u(x'', H'')$ with $x''\in\Sigma''_{-1}$ are both dense in a neighborhood of G. But no map in the former set can be Ω-conjugate to any map in the latter set.

References

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