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ON THE WEYL SPECTRUM OF A HILBERT SPACE OPERATOR

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ABSTRACT. Using the perturbation definition of the Weyl spectrum, conditions are given on a closed (possibly unbounded) linear operator T in a Hilbert space which allow the Weyl spectrum to be characterized as a subset of the spectrum of T.

1. **Introduction.** Let T be a closed linear operator with domain $D(T)$ dense in a Hilbert space H. Let $\sigma(T)$ denote the spectrum of T, $\pi_0(T)$ the set of eigenvalues of T, $\pi_{0f}(T)$ the set of eigenvalues of finite geometric multiplicity of T, and $\pi_{00}(T)$ the set of isolated eigenvalues of finite geometric multiplicity of T. (Here, "isolated" means isolated as points in $\sigma(T)$.) Thus

$$\pi_{00}(T) \subset \pi_{0f}(T) \subset \pi_0(T) \subset \sigma(T).$$

In 1909, H. Weyl [10] investigated the behavior of the spectrum of T under perturbation by compact operators and proved that if T is bounded and selfadjoint, then

$$(*) \qquad \bigcap \{\sigma(T+K) : K \text{ compact}\} = \sigma(T) - \pi_{00}(T).$$

For T an arbitrary closed linear operator, we denote the left-hand side of the above equation by $\omega(T)$ and call $\omega(T)$ the Weyl spectrum of T. Recently, several authors ([1]–[5], [7]) have proved that $\omega(T)=\sigma(T)-\pi_{00}(T)$ under conditions on T more general even than normality. All these authors except Bouldin assume that T is bounded with $D(T)=H$. In [3] and [4], Bouldin investigated several alternative definitions of the essential spectrum for a closed linear operator T and in particular gave conditions under which $(*)$ holds. In addition, he considered the effect of replacing the concept of geometric multiplicity by that of algebraic multiplicity.

In this note, we extend the results of [1] on bounded operators to the unbounded case. As in [1], we continue to use the classical definition of the

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Weyl spectrum given above rather than the currently more fashionable definition in terms of Fredholm operators. In §2, the nature of the Weyl spectrum $\omega(T)$ is investigated for an arbitrary closed linear operator T. In §3, a hypothesis on T is formulated which implies a modified form of $(*)$: $\pi_{00}(T)$ must be replaced by the set of isolated eigenvalues of finite algebraic multiplicity. A similar result in [**4**] is then obtained as a corollary. Finally, in §4, we give conditions under which $(*)$ is valid.

I would like to record here my appreciation to Richard Bouldin for stimulating correspondence and critical remarks.

2. **Some general properties of the Weyl spectrum.** Throughout this section T is a closed linear operator with domain $D(T)$ dense in a Hilbert space H.

We define the algebraic multiplicity of an isolated point $\lambda \in \sigma(T)$ as in [**3**]. For such a λ it is known that there exists a direct sum decomposition $H=A(\lambda)\oplus B(\lambda)$, each summand of which is invariant under $T-\lambda I$; the restriction N_λ of $T-\lambda I$ to $A(\lambda)$ is bounded and quasi-nilpotent and the restriction S_λ of $T-\lambda I$ to $B(\lambda)$ has a bounded inverse defined over all of $B(\lambda)$. The dimension $\dim A(\lambda)$ of $A(\lambda)$ is then by definition the algebraic multiplicity of λ. If λ is an eigenvalue of T, then the corresponding eigenspace $G(\lambda)$ is contained in $A(\lambda)$ so that the geometric multiplicity $\dim G(\lambda)$ never exceeds the algebraic multiplicity of λ.

We introduce a little more notation. Let $\alpha(T)$ denote the isolated eigenvalues of T of infinite algebraic multiplicity. Then let $\hat{\pi}_{0f}(T)=\pi_{0f}(T)-\alpha(T)$ and let $\hat{\pi}_{00}(T)=\pi_{00}(T)-\alpha(T)$. Thus, $\hat{\pi}_{00}(T)$ consists of the isolated eigenvalues of finite algebraic multiplicity.

LEMMA 2.1. $\sigma(T)-\pi_0(T)\subset\omega(T)$.

PROOF. The simple proof given in [**6**, Problem 143] for the case that T is bounded generalizes to the present situation.

LEMMA 2.2. $\sigma(T)-\pi_{0f}(T)\subset\omega(T)$.

PROOF. The proof in [**1**, Lemma 2] of the same result for T bounded generalizes with no essential change.

LEMMA 2.3. $\sigma(T)-\hat{\pi}_{0f}(T)\subset\omega(T)$.

PROOF. In view of Lemma 2.2, we need only show that $\lambda \in \omega(T)$ if λ is an isolated eigenvalue of infinite algebraic multiplicity but finite geometric multiplicity. For such a λ, it follows from [**8**, p. 240] that the range of N_λ is not closed. Hence, the range of $T-\lambda I$ is not closed. If $\lambda \notin \sigma(T+K)$ for some compact K, then $(T+K-\lambda I)^{-1}$ is a bounded operator with domain H, and so $(T+K-\lambda I)^{-1}K$ is compact. The theory of compact operators

[**9**, p. 279] then guarantees that $I-(T+K-\lambda I)^{-1}K$ has closed range. Using the factorization

$$T-\lambda I=[T+K-\lambda I][I-(T+K-\lambda I)^{-1}K],$$

we easily see that $T-\lambda I$ has closed range, a contradiction.

If T is bounded, we can give a completely elementary proof of this last lemma, without recourse to the theory of compact operators in [**9**] or [**8**, p. 240]. Assuming that $A(\lambda)$ is infinite dimensional, let $\{e_n\}$ be an infinite orthonormal sequence in $A(\lambda)$. Suppose $\lambda\notin\sigma(T+K)$ for some compact K; then $U=(T+K-\lambda I)^{-1}$ exists as a bounded operator on H. Let $E=(2\|U\|)^{-1}$ and choose $k\geqq 2$ so that $\|N_\lambda^k\|^{1/k}\leqq E$. This can be done since N_λ is quasi-nilpotent. Let $y_n=(T+K-\lambda I)^k e_n$. Then $y_n=(T-\lambda I)^k e_n+Ve_n$, where V is a sum of 2^k-1 operators each of which is a finite product of bounded operators at least one of which is the compact operator K. Thus V is compact. By passing to a subsequence if necessary, we may assume that $y=\lim Ve_n$ exists. Thus

$$\|y_n-y\|\leqq\|N_\lambda^k e_n\|+E^k\leqq 2E^k$$

for n sufficiently large. But $e_n=U^k y_n$ and hence

$$\|e_n-U^k y\|\leqq\|U\|^k\,\|y_n-y\|\leqq 2(E\,\|U\|)^k\leqq\tfrac{1}{2}$$

for n sufficiently large. Thus, if $m>n$,

$$\begin{aligned}2&=\|e_n\|^2+\|e_m\|^2=\|e_n-e_m\|^2\\&\leqq[\|e_n-U^k y\|+\|U^k y-e_m\|]^2\leqq 1,\end{aligned}$$

a contradiction.

If T is not bounded, the above proof would be valid if $A(\lambda)$ is invariant under K. Otherwise, the operator V might experience severe difficulties.

LEMMA 2.4. $\omega(T)\subset\sigma(T)-\hat{\pi}_{00}(T)$.

PROOF. Since $\omega(T)\subset\sigma(T)$, we need only show that $\omega(T)\cap\hat{\pi}_{00}(T)=\varnothing$. Suppose $\lambda\in\hat{\pi}_{00}(T)$. We look for a compact K for which $\lambda\notin\sigma(T+K)$. Since $1\leqq\dim A(\lambda)<\infty$, the operator K is defined by

$$\begin{aligned}Kx&=x,\quad\text{if } x\in A(\lambda),\\&=0,\quad\text{if } x\in B(\lambda)\end{aligned}$$

and extended by linearity to all of H is compact since its range is finite dimensional. Since N_λ is quasi-nilpotent, then $\sigma(N_\lambda)=\{0\}$ and it follows easily that $\sigma(N_\lambda+I)=\{1\}$. Thus $N_\lambda+I$ is one-to-one and onto. Since S_λ is also one-to-one and onto, it follows that $T+K-\lambda I$ is one-to-one and onto. Thus, $(T+K-\lambda I)^{-1}$ is a closed operator with domain H. By the closed graph theorem (see [**8**] or [**9**]), $(T+K-\lambda I)^{-1}$ is bounded and $\lambda\notin\sigma(T+K)$. Hence $\lambda\notin\omega(T)$.

3. **Weyl's theorem and algebraic multiplicity.** We use the generic phrase "Weyl's theorem" for any theorem which characterizes $\omega(T)$ as a subset of $\sigma(T)$. In this section, we give a sufficient condition that $\omega(T)=\sigma(T)-\hat{\pi}_{00}(T)$.

CONDITION C-1. If $\{\lambda_n\}$ is an infinite sequence of distinct points in $\hat{\pi}_{0f}(T)$, if $\lim \lambda_n=\lambda \in \hat{\pi}_{0f}(T)$, and if $\{x_n\}$ is a sequence of corresponding normalized eigenvectors, then the sequence $\{x_n\}$ does not converge.

We remark that Condition C-1 is slightly modified from the statement in **[1]**.

THEOREM 3.1. *If T satisfies* C-1, *then* $\omega(T)=\sigma(T)-\hat{\pi}_{00}(T)$.

PROOF. By Lemma 2.4, we must only show that $\sigma(T)-\hat{\pi}_{00}(T)\subset\omega(T)$. Now $\sigma(T)-\hat{\pi}_{00}(T)=[\sigma(T)-\hat{\pi}_{0f}(T)]\cup[\hat{\pi}_{0f}(T)-\hat{\pi}_{00}(T)]$. By Lemma 2.3, $\sigma(T)-\hat{\pi}_{0f}(T)\subset\omega(T)$. Since $\omega(T)$ is closed (topologically),

$$\mathrm{cl}(\sigma(T)-\hat{\pi}_{0f}(T))\subset\omega(T).$$

Thus, if suffices to show that $\lambda\in\omega(T)$ if $\lambda\in\hat{\pi}_{0f}(T)-\hat{\pi}_{00}(T)$ but $\lambda\notin \mathrm{cl}(\sigma(T)-\hat{\pi}_{0f}(T))$. Thus, there exists an infinite sequence $\{\lambda_n\}$ of distinct points in $\hat{\pi}_{0f}(T)$ which converges to λ. Let $\{x_n\}$ be a sequence of corresponding normalized eigenvectors. Then by Condition C-1, $\{x_n\}$ does not converge. Suppose now that $\lambda\notin\sigma(T+K)$ for some compact K. Then $(T+K-\lambda I)^{-1}$ exists as a bounded operator on H. Let

$$y_n=(T+K-\lambda I)x_n=(\lambda_n-\lambda)x_n+Kx_n.$$

By passing to a subsequence if necessary, we may assume $y=\lim Kx_n$ exists. Since $\lim \lambda_n=\lambda$, we have $\lim y_n=y$. But then

$$\lim x_n=\lim\,(T+K-\lambda I)^{-1}y_n=(T+K-\lambda I)y,$$

a contradiction.

If $\lambda\in\pi_0(T)$, Bouldin [4] says that the eigenspace $G(\lambda)$ corresponding to λ is not an asymptotic eigenspace if there exists a δ $(0<\delta<1)$ such that $|(x,y)|\leqq\delta$ if $x\in G(\lambda)$, $\|x\|=1=\|y\|$, and y is an eigenvector of T corresponding to some eigenvalue $\mu\neq\lambda$. We now get as a corollary to the above theorem a result of Bouldin **[4]**.

COROLLARY. *If each finite dimensional eigenspace of T is not an asymptotic eigenspace, then T satisfies Condition* C-1 *and, consequently,* $\omega(T)=\sigma(T)-\hat{\pi}_{00}(T)$.

PROOF. Suppose T does not satisfy Condition C-1. Then there exists an infinite sequence $\{\lambda_n\}$ of distinct points in $\hat{\pi}_{0f}(T)$ which converges to $\lambda\in\hat{\pi}_{0f}(T)$ and a sequence $\{x_n\}$ of corresponding normalized eigenvectors which converges. Let $x=\lim x_n$. Then $\lambda x=\lim \lambda_n x_n=\lim\, Yx_n$. Since T is closed, $x \in D(T)$ and $Tx=\lambda x$. Thus, $x \in G(\lambda)$ and clearly $\|x\|=1$. Therefore, $\lim(x, x_n)=(x, x)=1$ and $G(\lambda)$ is an asymptotic eigenspace.

4. **Weyl's theorem and geometric multiplicity.** We now give sufficient conditions in order that $\omega(T)=\sigma(T)-\pi_{00}(T)$. If T satisfies Condition C-1, we already know that $\omega(T)=\sigma(T)-\hat{\pi}_{00}(T)$. Thus, any condition which implies that $\hat{\pi}_{00}(T)=\pi_{00}(T)$ is of interest. We now consider

CONDITION C-2. If $\lambda \in \pi_{00}(T)$, then $T-\lambda I$ has closed range.

We remark that Condition C-2 above assumes less than the corresponding condition of [**1**].

THEOREM 4.1. *If T satisfies Condition* C-2, *then* $\hat{\pi}_{00}(T)=\pi_{00}(T)$.

PROOF. It suffices to show that $\pi_{00}(T)\subset\hat{\pi}_{00}(T)$. If $\lambda \in \pi_{00}(T)$, then λ is an isolated eigenvalue of finite geometric multiplicity and thus the null space of $T-\lambda I$ is finite dimensional. By Condition C-2, $T-\lambda I$ also has closed range. Thus, the null space of the quasi-nilpotent operator N_λ is finite dimensional and N_λ has closed range. It follows from [**8**, p. 240] that dim $A(\lambda)<\infty$.

COROLLARY. *If T satisfies both Conditions* C-1 *and* C-2, *then* $\omega(T)=\sigma(T)-\pi_{00}(T)$.

This corollary, even if T is bounded, is stronger than the result in [**1**]. Moreover, the results of Bouldin [**3**, Theorem 4 and Corollary 3] corresponding to Theorem 4.1 above are proved only in the case that T is bounded. In this case, the referee has pointed out that the condition (5) in [**3**, Corollary 3] is equivalent to dim $A(\lambda)<\infty$, which in turn is equivalent (by a theorem of Kato) to our Condition C-2. Thus, [**3**, Corollary 3] in conjunction with [**4**, Theorem 3] is, at least in the bounded case, essentially equivalent to our corollary above.

That the corollary above contains most known results is adequately documented in [**3**] and [**1**]. However, it is known [**5**] that $\omega(T)=\sigma(T)-\pi_{00}(T)$ holds for any Toeplitz operator. Although C-2 is vacuously satisfied in this case, Richard Bouldin has shown me an example (the adjoint of the unilateral shift on the Hardy space H_2) for which C-1 is violated. Thus, Weyl's theorem for Toeplitz operators lies deeper than any of the general theorems so far discovered.

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