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# LOCAL TRIVIALITY OF FIBERINGS

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ABSTRACT. We prove that a Hurewicz fibering or a Serre fibering is locally trivial if the total space is a connected separable metric ANR $n$-gm over a principal ideal domain and the base space is a weakly locally contractible paracompact finite dimensional space, and all fibers are homeomorphic to a space which is a connected 3-manifold with exactly one end and whose one point compactification is a 3-manifold and it has no false 3-cells, in particular a euclidean 3-space.

1. **Introduction.** A map $p:E\to B$ is a Hurewicz fiber map if $p$ has the covering homotopy property for all topological spaces and it is a Serre fiber map if $p$ has the covering homotopy property for polyhedrons. In **[10]**, Raymond conjectured that a Hurewicz fiber map is locally trivial if the total space $E$ is a manifold without boundary and the base space $B$ is a weakly locally contractible (wlc) paracompact space. In supporting the conjecture, Raymond proved that a Hurewicz fiber map is locally trivial if $E$ is a connected separable metric ANR (generalized) manifold (over a principal ideal domain) and $B$ is a wlc paracompact space and it has a fiber which contains a compact connected component of dimension $\leqq 2$.

The conjecture is false if $E$ is allowed to have nonempty boundary or if $E$ is not a manifold. The latter is due to the replacement theorem of Fadell, Langston and Tulley **[8]**. However, Kim extended the result of Raymond to the case where the manifold $E$ has nonempty boundary by imposing more conditions on the fibering, and to the case where the fibers are noncompact manifolds **[6]**. In particular, Kim proved that a Hurewicz fibering is locally trivial if all fibers are homeomorphic to either $R^1$ or $R^2$ (euclidean spaces) and conjectured that a Hurewicz fibering is locally trivial if all fibers are homeomorphic to $R^3$.

In this paper, we settle the above conjecture of Kim affirmatively. We actually prove a slightly more general theorem, as we have already done

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elsewhere in the case where the fibers are $R^2$. A similar thing can be said for a Serre fiber map. The method of proof is quite similar to the case where the fibers are $R^2$ [**6**]. We compactify the total space $E$ along each fiber and we use the results of Dyer and Hamstrom, namely the facts that a homotopically 2-regular map is a completely regular map when the dimension of fibers is low and a completely regular map is locally trivial when the dimension of fibers is low ([**2**], [**4**], and [**5**]).

In [7], Kim also extended the Dyer and Hamstrom result that a completely regular map is locally trivial to the case of higher dimensional fibers. However, we cannot extend our present result to the case where the fibers are $R^n$ $(n>3)$ because we do not know how to go by the first result of Dyer and Hamstrom to the case where the dimension of the fiber is big.

2. **Construction.** Let $p:E\to B$ be a Hurewicz fiber map from a connected separable metric ANR $n$-gm $E$ over a principal ideal domain onto a wlc paracompact finite (covering) dimensional base space $B$. By a generalized $n$-manifold ($n$-gm) we mean what Wilder and Raymond call a (locally orientable) cohomology $n$-manifold (see [**10**] and [**14**]). Suppose that all fibers are homeomorphic to a space $M$, where $M$ is a connected 3-manifold with exactly one end (see [**11**] for definition) and whose one point compactification is a 3-manifold. We note that $B$ is necessarily 0-connected and is a separable metric ANR [**3**], and $B$ is also locally compact because $E$ is locally compact and the map $p:E\to B$ is open [**3**].

Let $\bar{E}$ be the disjoint union of $E$ and the product space $B\times\{0\}$. We define $\bar{p}:\bar{E}\to B$ by $\bar{p}(e)=p(e)$ for each $e\in E$ and $\bar{p}(e)=b$ for each $e=(b,0)\in B\times\{0\}$. We give a topology $\overline{\mathscr{U}}$ on $\bar{E}$ in the following way: Let $\mathscr{U}$ be the collection of all open sets of $E$. Let $W$ be an open set of $B$ such that $p$ admits a cross section $f$ on $W$; furthermore there exists a closed subset $W_0$ of $p^{-1}(W)$ such that the closure of $W_0$ in $E$ is compact and $W_0\supset f(W)$ and $W_0\cap p^{-1}(b)$ is compact in $p^{-1}(b)$ for each $b\in W$. To see that such $W$ and $W_0$ exist, we use the local contractibility of $B$ and a fiber homotopy equivalence between $p^{-1}(W')$ and $W'\times p^{-1}(b_0)$ where $b_0\in W'$ and $W'$ is an open subset of $B$ which is uniformly contractible in $B$ (see [**6**, p. 59] for details). There are infinitely many such pairs $(W, W_0)$, $W\ni b$, for each point $b\in B$. The collection of all such pairs for all $b\in B$ will be denoted by $\mathscr{C}$. Let $\mathscr{V}'$ be the collection of sets of the form $\bar{p}^{-1}(W)-W_0$ for each $(W, W_0)\in\mathscr{C}$, and let $\mathscr{V}$ be the collection of all subsets $V$ of $\bar{E}$ such that $V\cap E$ is open in $E$ and $V\cap(\bar{E}-E)\neq\varnothing$, and for each $x\in V\cap(\bar{E}-E)$ there exists an element $V'\in\mathscr{V}'$ such that $x\in V'\subset V$. Then any element $V'$ of $\mathscr{V}'$ is an element of $\mathscr{V}$ (see [**6**]). Now let $\overline{\mathscr{U}}$ denote the collection of all elements of $\mathscr{U}$ and $\mathscr{V}$. Then it is easy to see that $\overline{\mathscr{U}}$ is a topology on $\bar{E}$. Henceforth $\bar{E}$ denotes a topological space with the topology $\overline{\mathscr{U}}$.

2.1. LEMMA. (i) *The map* $\bar{p}:\bar{E}\to B$ *is continuous and open,*

(ii) *the subspace* $B\times\{0\}$ *of* $\bar{E}$ *is homeomorphic to* $B$,

(iii) *the space* $\bar{E}$ *is a locally compact Hausdorff space,*

(iv) *the space* $\bar{E}$ *is a topologically complete metric space,*

(v) *the space* $\bar{p}^{-1}(b)$, *as a subspace of* $\bar{E}$, *is the one point compactification of* $p^{-1}(b)$ *for each* $b\in B$, *hence* $\bar{p}^{-1}(b)$ *is a compact 3-manifold with added point as its interior point,*

(vi) *the map* $\bar{p}:\bar{E}\to B$ *is a proper map.*

PROOF. Some of these properties are direct consequences of the definition of the space $\bar{E}$ and the map $\bar{p}:\bar{E}\to B$ and nontrivial facts were proved in [**6**, p. 60] (see also [**12**]).

We recall here the definition of the homotopic $n$-regularity of a map. A map $f$ from a metric space $X$ onto a metric space $Y$ is homotopically $n$-regular if it is open and proper and if for given $x\in X$ and $\varepsilon>0$ there exists $\delta>0$ such that each mapping of a $k$-sphere $S^k$, $k\leqq n$, into $S(x,\delta)\cap f^{-1}(y)$, $y\in Y$, is homotopic to a constant map in $S(x,\varepsilon)\cap f^{-1}(y)$, where $S(x,\varepsilon)$ is the $\varepsilon$-neighborhood of $x$.

2.2. LEMMA. *The map* $\bar{p}:\bar{E}\to B$ *is homotopically 2-regular.*

PROOF. By (2.5) of [**10**], $p:E\to B$ is a homotopically 2-regular map without assuming the properness. Therefore, it suffices to verify the conditions of the homotopic 2-regularity of $\bar{p}$ on the points in $\bar{E}-E$. But this follows by our definition of the neighborhood systems of points in $\bar{E}-E$ which are defined "fiber-wise" in a sense. More specifically, let a point $x\in(\bar{E}-E)$ and $\varepsilon>0$ be given. Since $S(x,\varepsilon)$ is open in $\bar{E}$, it contains an element $V\in\mathscr{V}$ which is an open neighborhood of $x$ such that $V=\bar{p}^{-1}(W)-W_0$ for an element $(W,W_0)\in\mathscr{C}$, where $\bar{p}(x)\in W$. Since $\bar{p}^{-1}(\bar{p}(x))\cap V$ is open in $\bar{p}^{-1}(\bar{p}(x))$ and $x$ is an interior point of a 3-manifold by Lemma 2.1, there exists a closed 3-cell $D_x$ of a positive diameter around $x$ in $\bar{p}^{-1}(\bar{p}(x))$ such that $D_x\subset V$; i.e., $D_x=S(x,\alpha)\cap\bar{p}^{-1}(\bar{p}(x))$ where $\alpha$ is a number. Furthermore, $V$ is an open neighborhood of $y=\bar{p}^{-1}(c)\cap(\bar{E}-E)$ and $\bar{p}^{-1}(c)\cap V$ is open in $\bar{p}^{-1}(c)$ for each $c\in W$. Therefore, we can find a neighborhood $U$ of $x$ in $\bar{E}-E$ so that for each $y\in U$, there exists a closed 3-cell $D_y$ around $y$ in $\bar{p}^{-1}(\bar{p}(y))$ whose diameter is bigger than some number $\alpha>0$ and $D_y\subset V$. If $\alpha$ is once chosen we may as well assume that the distance from $x$ to the frontier of $U$ denoted by $d(x,\mathrm{Fr}(U))$ is less than $\alpha/4$ by changing $U$ if necessary. Now since $\bar{p}(U)$ is again open in $B$, $d(x,\bar{p}^{-1}(\bar{p}(\mathrm{Fr}(U))))=\beta$ is nonzero. Take $\delta$ to be the minimum number of $\beta$ and $\alpha/4$. Then the open neighborhood $S(x,\delta)$ of $x$ is contained in the union of all $D_y$, $y\in U$, hence in $V$. For if $z\in S(x,\delta)$ then $d(x,z)<\delta\leqq\beta$. Therefore, first of all $z\in\bar{p}^{-1}(\bar{p}(U))$; i.e., $\bar{p}^{-1}(\bar{p}(z))\cap(\bar{E}-E)=y\in U$.

Furthermore, $d(y, z) \leqq d(y, x)+d(x, z)<\alpha/4+\alpha/4=\alpha/2$, and hence $z \in \operatorname{Int}(D_y)$.

Then if $S(x, \delta)\cap\bar{p}^{-1}(c)$ is nonempty, any map $g: S^k \to S(x, \delta)\cap\bar{p}^{-1}(c)$ is homotopic to a constant map in $S(x, \varepsilon)$ for all $k$ since $S(x, \delta)\cap\bar{p}^{-1}(c)\subset \operatorname{Int}(D_y)\subset V\subset S(x, \varepsilon)$ and $D_y$ is a 3-cell, where $y \in (\bar{E}-E)$ with $\bar{p}(y)=c$ in $B$.

We know that $\bar{p}$ is an open and proper map by Lemma 2.1. Therefore $\bar{p}$ is a homotopically 2-regular map (in fact, $n$-regular).

We prove another lemma which is independent of the above construction.

2.3. LEMMA. *Let $M$ be a 3-manifold whose one point compactification $M^+$ is again a 3-manifold. If all homotopy 3-cells in $M$ are 3-cells then all homotopy 3-cells in $M^+$ are 3-cells too.*

PROOF. It suffices to consider a homotopy 3-cell $D$ in $M^+$ such that $x \in \operatorname{Int}(D)$. Choose an arc $A$ from $x$ to a point of the boundary of $D$, and a nice tabular neighborhood $N$ of the arc $A$ which is a real 3-cell. Then $D-\operatorname{Int}(N)$ is a homotopy 3-cell in $M$. Therefore, $D-\operatorname{Int}(N)$ is a real 3-cell. On the other hand, there is an isotopy of $D$ keeping all of it fixed except for a neighborhood of $N$ so that $D-\operatorname{Int}(N)$ image is $D$ itself; i.e., $D-\operatorname{Int}(N)$ is homeomorphic to $D$ itself. Therefore $D$ is a real cell.

## 3. Main theorems.

3.1. THEOREM. *Let $p: E\to B$ be a Hurewicz fiber map from a connected separable metric ANR $n$-gm $E$ over a principal ideal domain onto a wlc paracompact finite (covering) dimensional base space $B$. Suppose that all fibers are homeomorphic to a space $M$ where $M$ is a connected 3-manifold with exactly one end and whose one point compactification is a 3-manifold and all homotopy 3-cells in $M$ are 3-cells. Then the fibering $(E, B, p)$ is locally trivial. If $B$ is contractible then it is a product fiber space.*

This theorem implies our original problem as a special case.

COROLLARY 1. *Let $p: E\to B$ be a Hurewicz fiber map from a connected separable metric ANR $n$-gm $E$ over a principal ideal domain onto a wlc paracompact finite (covering) dimensional space $B$. If all fibers are homeomorphic to $R^3$, then the fibering $(E, B, p)$ is locally trivial. If $B$ is contractible, then it is a product fiber space.*

PROOF OF THEOREM 3.1. We note again that $B$ is necessarily 0-connected, locally compact, separable metric, and ANR. Let $\bar{E}$ be the disjoint union of $E$ and the product space $B\times\{0\}$ and $\bar{p}: \bar{E}\to B$ be defined by $\bar{p}(e)=p(e)$ for each $e \in E$ and $\bar{p}(e)=b$ for each $e=(b, 0)\in B\times\{0\}$. Let $\mathcal{U}$ be the topology for $\bar{E}$ that is defined in §2. By Lemma 2.1, $\bar{E}$ is a topologically complete metric space. Therefore there exist a complete metric space $E'$ and a homeomorphism $h$ between $E'$ and $\bar{E}$. If we define a map $p':E'\to B$ to be $\bar{p}\cdot h$, then it is a homotopically 2-regular map from a complete metric space $E'$ onto a finite (covering) dimensional metric space $B$ such that each inverse under $p'$ is homeomorphic to a compact 3-manifold with boundary by Lemmas 2.1 and 2.2. Since each fiber $p^{-1}(b)$ has no homotopy 3-cells which are not real 3-cells, the space $\bar{p}^{-1}(b)$ that is the one point compactification of $p^{-1}(b)$ has no homotopy 3-cells which are not real 3-cells by Lemma 2.3. Therefore by (6.1) of [**4**], $p':E'\to B$ is locally trivial. That is, for each $b\in B$, there exist an open set $U$ of $b$ in $B$ and a homeomorphism $h'_U: U\times p'^{-1}(b)\to p'^{-1}(U)$ such that the diagram

$$\begin{tikzcd} U\times p'^{-1}(b) \arrow[r, "h'_U"] \arrow[dr, "\pi"'] & p'(U) \arrow[d, "p'|_{p'^{-1}(U)}"] \\ & U \end{tikzcd}$$

commutes, where $\pi$ is the projection map. Then the commutative diagram below gives us a trivialization of $\bar{p}|_U:\bar{p}^{-1}(U)\to U$:

$$\begin{tikzcd} U\times h^{-1}(\bar{p}^{-1}(b)) \arrow[r] \arrow[d, "\mathrm{id}\times h^{-1}"] \arrow[dr, "\pi"] & h^{-1}(\bar{p}^{-1}(U)) \arrow[r] \arrow[d, "p'|_{p'^{-1}(U)}"] & \bar{p}^{-1}(U) \arrow[dl, "\bar{p}|_{\bar{p}^{-1}(U)}"] \\ U\times \bar{p}^{-1}(b) \arrow[r, "\pi"'] & U & \end{tikzcd}$$

Therefore $\bar{p}:\bar{E}\to B$ is locally trivial. From this fact we can conclude that $p:E\to B$ itself is locally trivial. Since the proof of this is exactly the same as the one of the case where the dimension of the fiber is two [**6**, p. 63], we only give a sketch of the proof rather than a detailed proof.

Let

$$\begin{tikzcd} U\times F \arrow[r, "h"] \arrow[dr, "\pi"'] & \bar{p}^{-1}(U) \arrow[d, "\bar{p}|_{\bar{p}^{-1}(U)}"] \\ & U \end{tikzcd}$$

be a local trivialization of $\bar{p}:\bar{E}\to B$, where $F$ is the fiber. Suppose the image of $U\times(F-p^{-1}(b))$ under $h$ is not equal to $\bar{p}^{-1}(U)-p^{-1}(U)$. (If it is equal, then nothing is to be done.) Define $f:U\to U\times F$ by $f(c)=(c, x_0)$, $c\in U$, $x_0=F-p^{-1}(b)$, and $g:U\to\bar{p}^{-1}(U)$ by $g(c)=\bar{p}^{-1}(c)-p^{-1}(c)$, $c\in U$. Then $h^{-1}g: U\to U\times F$ is different from $f$. We find an open set $V$ in $U$ and a fiber preserving homeomorphism $\phi: V\times F\to V\times F$ such that $\phi(h^{-1}g(V))=f(V)$. Here we use the fact that the group of homeomorphisms of an $n$-disk $D^n$ fixing $\partial D^n$ and an interior point pointwise is locally contractible for all $n$ [**1**]. Then

$$\begin{tikzcd} V\times F \arrow[r, "h'=h|_{V\times F}"] \arrow[dr, "\pi"'] & \bar{p}^{-1}(V) \arrow[d, "\bar{p}|_{\bar{p}^{-1}(V)}"] \\ & V \end{tikzcd}$$

commutes. Therefore, $h'\phi^{-1}:(V\times F, f(V))\to(V\times F, h^{-1}g(V))\to(\bar{p}^{-1}(V), g(V))=(\bar{p}^{-1}(V), \bar{p}^{-1}(V)-p^{-1}(V))$ is a homeomorphism. Hence $h_V= h'\phi^{-1}|_{V\times F-V\times(\bar{p}^{-1}(b)-p^{-1}(b))}$ is a homeomorphism such that the diagram

$$\begin{tikzcd} V\times p^{-1}(b) \arrow[rr, "h_V"] \arrow[dr, "\pi"'] & & p^{-1}(V) \arrow[dl, "p|_{p^{-1}(V)}"] \\ & V & \end{tikzcd}$$

commutes. This is a local trivialization of the map $p:E\to B$. The last statement of the theorem is a result of the general bundle theory. Q.E.D.

We state an analogue theorem in Serre fibering.

3.2. THEOREM. *Let $p:E\to B$ be a Serre fiber map from a connected separable metric ANR n-gm E over a principal ideal domain onto a wlc paracompact finite* (*covering*) *dimensional base space B. Suppose that all fibers are homeomorphic to a space M where M is a connected* 3-*manifold with exactly one end and whose one point compactification is a* 3-*manifold and all homotopy* 3-*cells in M are* 3-*cells. Then the fibering* $(E, B, p)$ *is locally trivial. If B is contractible then it is a product fiber space.*

PROOF. By Theorem 2 of [**9**], $p$ is homotopically $n$-regular; then by Ungar's theorem [**13**], $p$ is a Hurewicz fiber map Therefore the local triviality of the map follows by Theorem 3.1.

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