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THE LOCATION OF THE ZEROS OF THE DERIVATIVES OF FABER POLYNOMIALS

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ABSTRACT. We prove a property of Faber polynomials which supports a conjecture concerning the zeros of Faber polynomials.

1. **Introduction.** Let K be a compact set consisting of more than one point having a connected and simply connected complement, $\tilde{K}$. Let $f(z)$ be the schlicht function mapping $\tilde{K}$ onto the exterior of the circle $\Sigma_w = \{w : |w| = 1\}$, normalized so that

$$f(z) = z + a_0 + a_{-1}z^{-1} + \cdots. \tag{1}$$

This normalization can always be achieved by changing units appropriately in the z plane, and we assume that this is done.

The polynomial part of $(f(z))^n$, $F_n(z)$, is called the Faber polynomial of degree n associated with K. There is a conjecture that for each n, $n=1, 2, \cdots$, the zeros of $F_n(z)$ lie in $H(K)$, the convex hull of K. I will show in this paper that $H(K)$ contains all the zeros of $F_n'(z)$, $n=1, 2, \cdots$.

2. Let

$$z = g(w) = w + b_0 + b_{-1}w^{-1} + \cdots$$

be the inverse function of $w=f(z)$. This is a schlicht function mapping the exterior of Σ_w onto $\tilde{K}$. Since $g(w)$ is bounded in $\{w : 1 < |w| < R\}$, $R > 1$, $\lim_{r\to 1^+} g(re^{i\theta})$ exists for almost all θ, $0 \leqq \theta \leqq 2\pi$. Let $g(\theta)$ be a function defined as this limit, where it exists, and as a value lying in K for other values of θ. Then, because of the schlichtness of $g(w)$, all the values of $g(\theta)$ lie in K. Finally, using Lebesgue's theorem of bounded convergence, we find that

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} g(\theta)e^{-in\theta}\,d\theta &= b_n, & n &= 0, -1, \cdots, \\ &= 1, & n &= 1, \\ &= 0, & n &= 2, 3, \cdots. \end{aligned} \tag{2}$$

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3. Faber [1] showed that

$$\frac{g'(w)}{g(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}} \tag{3}$$

and from this we easily obtain

$$\frac{1}{g(w) - z} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{nw^n}. \tag{4}$$

4. Our object now is to get another representation for $F_n'(z)$ by regarding $(g(w)-z)^{-1}$ as the reciprocal of a power series, namely

$$\frac{1}{g(w) - z} = \frac{1}{w + (b_0 - z) + b_{-1}w^{-1} + \cdots}. \tag{5}$$

For a power series $\sum_{n=0}^{\infty} c_n z^n$, it is readily found that if $\sum_{n=0}^{\infty} d_n z^n = (\sum_{n=0}^{\infty} c_n z^n)^{-1}$, then

$$d_n = (-1)^n \frac{\begin{vmatrix} c_1 & c_0 & 0 & \cdot \\ c_2 & c_1 & c_0 & 0\cdot \\ & & & c_0 \\ c_n & & & c_1 \end{vmatrix}}{c_0^{n+1}}. \tag{6}$$

We can apply (6) to (5) and use (4) to obtain

$$\frac{F_{n+1}'(z)}{n+1} = (-1)^n \begin{vmatrix} b_0 - z & 1 & 0 & & \cdot \\ b_{-1} & b_0 - z & 1 & 0 & \cdot \\ & & & & 1 \\ b_{-(n-1)} & & & b_0 - z & \end{vmatrix}. \tag{7}$$

Thus we see that the zeros of the derivatives of Faber polynomials are of the eigenvalues a type of matrix known as a Toeplitz matrix.

5. If $(e_{p,q})$ is an nth order matrix and if λ is an eigenvalue, then there is a nondegenerate vector $(x_1, \cdots, x_n)$ such that

$$\sum_{q=1}^{n} e_{p,q} x_q = \lambda x_p, \qquad p = 1, \cdots, n.$$

From this we obtain

$$\sum_{p=1}^{n}\sum_{q=1}^{n} e_{p,q}x_q\bar{x}_p = \lambda\sum_{p=1}^{n}|x_p|^2.$$

Now, in the case of (7), we have $e_{p,q}=b_{q-p}$, if we understand that $b_1=1$ and $b_n=0$ for $n=2, 3, \cdots$. Thus if z_0 is a zero of $F'_{n+1}(z)$ we find that there is a nondegenerate vector $(x_1, \cdots, x_n)$ such that

$$\sum_{p=1}^{n}\sum_{q=1}^{n} b_{q-p}x_q\bar{x}_p = z_0\sum_{p=1}^{n}|x_p|^2.$$

Using (2) this becomes

$$\frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,|x_1 + x_2e^{-i\theta} + \cdots + x_ne^{-i(n-1)\theta}|^2\,d\theta$$
$$= z_0\frac{1}{2\pi}\int_0^{2\pi}|x_1 + x_2e^{-i\theta} + \cdots + x_ne^{-i(n-1)\theta}|^2\,d\theta.$$

This shows that z_0 is a weighted arithmetic average of the values in the essential range of $g(\theta)$, and thus lies in the convex hull of the essential range of $g(\theta)$. We have already noted that the values of $g(\theta)$ lie in K, and thus the convex hull of the essential range of $g(\theta)$ lies in the convex hull of K, and our proof that z_0, a zero of $F'_{n+1}(z)$, lies in the convex hull of K is complete. The results of this paper now permit us to apply the results of [2] to the zeros of the derivatives of Faber polynomials. In addition, the methods of [2] can be applied to obtain results on the asymptotic distribution of the Faber polynomials themselves.

References

1. G. Faber, *Über polynomische Entwickelungen*, Math. Ann. **57** (1903), 389–408.

2. J. L. Ullman, *Toeplitz matrices associated with a semi-infinite Laurent series*, Proc. London Math. Soc. **22** (1971), 164–192.

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