

ON TOPOLOGICAL PROPERTIES OF SETS ADMITTING VARISOLVENT FUNCTIONS

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ABSTRACT. Mairhuber's theorem on Haar subspaces is generalized for the nonlinear case, where varisolvent functions are considered.

1. According to a well-known theorem of Mairhuber [4] a compact set in \mathbf{R}^N is homeomorphic to a subset of a circumference Γ , if it admits a real Haar subspace with dimension $N \geq 2$. This result has been proved for general compact spaces by Curtis [2] and by Sieklucki [7]. A first extension of Mairhuber's theorem to nonlinear families of functions was given by Dunham [3]. In this note we will derive a stronger result.

THEOREM 1. *Let the compact set Q admit a varisolvent family of functions. If the degree of solvence is bounded and if the maximal degree is greater than 1, then Q is homeomorphic to a subset of a circumference Γ . Moreover, if the degree is an even number at some element, then the subset must be proper.*

As a consequence of this theorem it is natural to consider varisolvent functions only on intervals in \mathbf{R} . This is the specialization used throughout the literature. The principal part of the proof will treat the case where the degree n is 2. Afterwards, the induction for $n > 2$ proceeds as in the proofs for linear families presented in [6], [8, p. 218].

2. We define varisolvent functions on a compact set Q . Let the real function $F(a, x)$ be defined for $x \in Q$ and $a \in P$, where P is the parameter space [5]. For all $a \in P$ we assume $F(a, \cdot) \in C(Q)$, but there is no need to endow P with a topology.

DEFINITION. (i) F has Property Z of degree m at $a^* \in P$, if for any $a \neq a^*$, $F(a, x) - F(a^*, x)$ has at most $m-1$ zeros for $x \in Q$.

(ii) F is solvent of degree m at $a^* \in P$ if given a set of m distinct points $x_j \in Q, j=1, 2, \dots, m$, and $\varepsilon > 0$, then there exists a

$$\delta = \delta(a^*, \varepsilon, x_1, x_2, \dots, x_m)$$

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such that $|F(a^*, x_j) - y_j| < \delta$ implies the existence of a parameter $a \in P$, satisfying

$$(1) \quad F(a, x_j) = y_j, \quad j = 1, 2, \dots, m,$$

as well as $\|F(a, \cdot) - F(a^*, \cdot)\| < \varepsilon$.

(iii) F is varisolvent if F possesses both the properties (i) and (ii) with the same degree for each $a \in P$.

At first we prove

LEMMA 2. Let Γ be a circumference and q a point outside Γ . If $F(a, x)$ is varisolvent on Γ (resp., on $\Gamma \cup \{q\}$), then the degree of solvence is odd (resp., at most 1) at each $a \in P$.

PROOF. Let $m = m(a^*)$ be even. Select m distinct points $x_i \in \Gamma$ in cyclic order. For sufficiently small $\delta > 0$ there exists a solution of

$$(2) \quad F(a, x_i) = y_i, \quad i = 1, 2, \dots, m,$$

where $y_i = F(a^*, x_i) + (-1)^i \delta$. Obviously, $F(a, x) - F(a^*, x)$ has m zeros on Γ , contradicting Property Z. This completes the proof for the statement on Γ . Concerning $\Gamma \cup \{q\}$, the statement for even m is a consequence of the preceding one. It is sufficient to consider odd degrees $m = m(a^*)$. Here, select $m-1$ distinct points $x_i \in \Gamma$ in cyclic order. Define $y_i, i = 1, 2, \dots, m-1$, as above and set $y_0 = F(a^*, q)$. Then the solution of

$$(3) \quad \begin{aligned} F(a, x) &= y_i, & i &= 1, 2, \dots, m-1, \\ F(a, q) &= y_0 \end{aligned}$$

yields a contradiction to Property Z. \square

3. Before proving the main theorem we will recall the linear case. Let u_1, u_2 be a base of a two dimensional Haar subspace. Then

$$x \rightarrow \{u_1(x), u_2(x)\}$$

defines an injective continuous mapping of Q into the real projective line. This mapping was considered in [8, p. 221]. If $u_2(x)$ has no zero in Q , we may use the mapping $\varphi: Q \rightarrow \mathbf{R}$ which sends x to $u_1(x)/u_2(x)$. We may even abandon the assumption on $u_2(x)$. By virtue of a lemma of Schoenberg and Yang (to be mentioned below) it is only necessary to present a mapping of $Q \setminus U$ into Γ for each open set $U \subset Q$. For given U choose a function u_2 without zero in $Q \setminus U$. Moreover, φ may be defined by the relation $\varphi(x) \cdot u_2(x) = u_1(x)$. This concept will be translated to the nonlinear case.

4. PROOF OF THEOREM 1. Let F be a varisolvent function on Q . We shall proceed by induction on n , the number of maximal degree of solvence. Choose $a^* \in P$ with $m(a^*) = n$. Observe that the translation

$\tilde{F}(a, x) = F(a, x) - F(a^*, x)$ generates a function \tilde{F} , which has the same degrees of Property Z and solvence as F and which is also varisolvent. Thus we may assume $F(a^*, x) \equiv 0$ without loss of generality.

At first let $n=2$ and let U be an arbitrary nonvoid open set in Q . Select $q_1 \in U$ and $q_2 \in M := Q \setminus U$. From the definition of solvence we know that there exists a $\delta > 0$ and a mapping from the unit square in R^2 into $C(Q)$:

$$(4) \quad (y_1, y_2) \rightarrow F(a(y_1, y_2), \cdot),$$

such that

$$F(a(y_1, y_2), q_i) = \delta \cdot y_i, \quad i = 1, 2.$$

Since $m(a^*)$ is maximal, it follows from Property Z that the mapping is uniquely defined for fixed δ . By virtue of Theorem 1 in [1] the mapping is continuous.

For $-1 \leq t \leq +1$ set $a_t = a(0, t)$. Obviously, we have $F(a_0, \cdot) = F(a^*, \cdot)$ and $F(a_t, q_1) = 0$. Hence, $m(a_t) = 2$ and Property Z implies

$$(5) \quad F(a_t, x) \neq F(a_s, x) \neq 0, \quad x \in M,$$

for all $s, t \in [-1, +1]$, $s \neq t \neq 0$. Since M is compact, it follows that

$$(6) \quad \min_{x \in M} \{|F(a_1, x)|, |F(a_{-1}, x)|\} = \eta > 0.$$

Furthermore, from the solvence property at a^* we conclude that there exists a parameter $b \in P$ such that

$$(7) \quad F(b, q_1) \neq 0$$

and

$$(8) \quad \|F(b, \cdot)\| < \eta.$$

Now we are ready to define a mapping $\varphi: M \rightarrow [-1, +1]$ by

$$(9) \quad F(a_{\varphi(x)}, x) = F(b, x), \quad x \in M.$$

We claim that the mapping is well defined. Indeed, by virtue of (4), (5) and (6) we have

$$F(a_1, x) \geq \eta \quad \text{and} \quad F(a_{-1}, x) \leq -\eta,$$

or

$$F(a_{-1}, x) \geq \eta \quad \text{and} \quad F(a_1, x) \leq -\eta,$$

for each $x \in M$. Hence, for each $x \in M$ the relation $F(a_t, x) = F(b, x)$ holds for at least one $t \in [-1, +1]$. By virtue of (5), this value is unique. Furthermore, φ is injective. If $\varphi(x_1) = \varphi(x_2) = t$, then $F(a_t, \cdot) - F(b, \cdot)$ has two zeros, which implies $F(a_t, \cdot) \equiv F(b, \cdot)$, contradicting $F(b, q_1) \neq F(a_t, q_1) = 0$. Finally, from the continuity of the mapping $[-1, +1] \times M \ni (t, x) \rightarrow F(a_t, x) - F(b, x)$, from compactness, and from the injectivity

of φ , we conclude that φ is continuous. Hence, $M=Q\setminus U$ is homeomorphic to a subset of \mathbf{R} or a subset of Γ . Since this holds for each open set U , from Lemma 1 in [6] (the same as Lemma 2.4 in [8, p. 219]), we obtain that Q can be homeomorphically embedded into Γ , or Q is homeomorphic to a union of Γ and a single point $q \notin \Gamma$. The second case is excluded by virtue of Lemma 2. Thus, the statement for $n=2$ is established.

Now, let $n>2$. Assume that Theorem 1 is true for $n-1$. Let U be an arbitrary open set in Q . Choose $q \in U$ and $a^* \in P$ with $m(a^*)=n$. Set

$$P' = \{a \in P : F(a, q) = F(a^*, q)\}.$$

Obviously, $F(a, x)$ is varisolvant for $x \in Q\setminus U$ and $a \in P'$. By this restriction the degree of solvence is reduced by one for each $a \in P'$. Since $a^* \in P'$, the induction hypothesis assures that $Q\setminus U$ is homeomorphic to a subset of Γ .

By the same arguments as those used for the case $n=2$, the proof of the theorem is completed. \square

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